Partial Differential Equations

of Second and Higher Degrees

&

Spherical Harmonics

FOR

POST-GRADUATE AND HONS. STUDENTS

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PREFACE

This book is a revised edition of the previous one. It embodies the lectures given in classes by the authors.

An effort has been made to put the subject matter in a clear and simple language. Obscure points which are a source of trouble have been thoroughly explained.

The authors shall feel rewarded if the book is found useful by students and teachers.

THE AUTHORS

PREFACE TO THE THIRD EDITION

The book has been thoroughly revised and enlarged.

Shorter and easier methods have been inserted in place of those difficult and lengthy.

Two more chapters rendered necessary by recent examination trends have been added in Spherical Harmonics.

Difficult questions have been solved or indicated.

We greatly thank our publisher and the printers for the nice printing of the book.

September 1963 THE AUTHORS

DEDICATION

Dedicated with feelings of devotion and deep affection to the ever-inspiring and ever-fresh memory of our **Great Teacher Dr. Ganesh Prasad D. Sc.,** who, besides being one of the great Mathematicians of the world, was a classical example of Purity, Fearlessness, Truth and Simplicity and was one of the great Builders of the subject of Spherical Harmonics. He discovered flaws in the arguments and writings of so many of the greatest mathematicians of the world both of the past and of his own time.

Being an extra-ordinary genius and a person of supremely high character, he was deeply loved and universally respected. He was the noblest and purest of men. He devoted his life to the service and welfare of the student community.

On 9th March 1935, at Agra, he left this world for his Permanent Abode in Heaven. He was born in Balia, educated at Balia, Allahabad, Cambridge and Gottingen (Germany). At the time of his death, he was the Hardinge Professor of Pure Mathematics in the Calcutta University, a position of the highest honour and distinction in Mathematics in India. May his soul rest in peace ! May his life ever inspire us on the path of knowledge, steadfastness and virtue ! May God give us capacity and courage to follow his foot-prints on the wide sands of life ! May the 'Kindly Light' lead us.

DHARMA VIRA
S. C. MITTAL

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Partial Differential Equations

Spherical Harmonics

CHAPTER I

PARTIAL DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDERS

1·0. The general linear equations of an order higher then the first.

A partial differential equation, which is linear with respect to the dependent variable and its partial differential coefficients, and in which the coefficients are constants, is called a linear partial differential equation with constant coefficients. It is of the form

$$\frac{\partial^n z}{\partial x^n}+A_1 \frac{\partial^n z}{\partial x^{n-1} \partial y}+A_2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2}+\dots$$

$$+A_n \frac{\partial^n z}{\partial y^n}+B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}}+\dots$$

$$+M \frac{\partial z}{\partial x}+N \frac{\partial z}{\partial y}+Pz=f(x, y). \qquad \dots(1)$$

On denoting $\frac{\partial}{\partial x}$ by D and $\frac{\partial}{\partial y}$ by D', it can be written as

$$(D^n+A_1 D^{n-1} D'+\dots+A_n D'^n+\dots MD+ND'+P)\ z =f(x, y) \qquad \dots(2)$$

or

$$\phi(D, D')\ z=f(x, y). \qquad \dots(3)$$

1·1. As in case of ordinary linear differential equations, the complete solution of (3) will consist of two parts, *the complementary functions* and *particular integral*, the complementary function being the solution of

$$\phi(D, D')\ z=0.$$

1·11. Complementary functions of homogeneous equations with constant coefficient.

This is the simplest case of the general equation and in it all the differential coefficients that occur are of the nth order; so it may be written as

$$(D^n+A_1D^{n-1}D'+A_2D^{n-2}D'^2+\ldots+A_nD'^n)\,z=V$$

or $$(D-\alpha_1D')\,(D-\alpha_2D')\ldots(D-\alpha_nD')\,z=V,$$

where $\alpha_1, \alpha_2\ldots\alpha_n$ are the roots of the auxiliary equation

$$m^n+A_1m^{n-1}+A_2m^{n-2}\ldots+A_n=0.$$

Now considering $(D-\alpha_rD')\ z=0$,

i.e. $$\frac{\partial z}{\partial x}-\alpha_r\frac{\partial z}{\partial y}=0, \text{ or, } p-\alpha_rq=0,$$

its subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-\alpha_r}=\frac{dz}{0}$$

or $$\alpha_rx+y=c_1 \text{ and } z=c_2.$$

$\therefore\ z=F_r\,(y+\alpha_rx)$ is the solution of $(D-\alpha_rD')=0$.

Hence as in case of ordinary linear equation the complementary function is

$$z=F_1\,(y+\alpha_1x)+F_2\,(y+\alpha_2x)\ldots+\ldots+F_n\,(y+\alpha_nx).$$

1·12. If the auxiliary equation has equal roots.

Consider the equation

$$(D-mD')^2\,z=0. \qquad \ldots(1)$$

Now let $$(D-mD')\,z=u. \qquad \ldots(2)$$

$\therefore$ (1) becomes $(D-mD')\,u=0$,

so that $$u=F\,(y+mx).$$

$$\therefore\ (D-mD')\,z=F\,(y+mx)\ \text{[from (2)]}$$

or, $$\frac{\partial z}{\partial x}-m\frac{\partial z}{\partial y}=F\,(y+mx).$$

Its subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{F\,(y+mx)}.$$

$$\therefore \quad y+mx=c_1. \qquad \ldots(1)$$

And then, $$\frac{dx}{1}=\frac{dz}{F(c_1)}.$$

$$\therefore \quad z-xF(c_1)=c_2$$

or $$z-xF(y+mx)=c_2. \qquad \ldots(2)$$

$\therefore$ the solution is,

$$\phi\,[z-xF(y+mx),\ y+mx]=0 \qquad \text{[from (1) and (2)]}$$

or $$z-xF(y+mx)=F_1(y+mx), \text{ giving } z.$$

In general the solution of

$$(D-mD')^r z=0$$

is $$z=F_1(y+mx)+xF_2(y+mx)+\ldots+x^{r-1}F_r(y+mx).$$

Examples 1 (1)

1. *Solve*

$$\frac{\partial^2 z}{\partial x^2}-3\frac{\partial^2 z}{\partial x\,\partial y}+2\frac{\partial^2 z}{\partial y^2}=0.$$

The auxiliary equation is

$$m^2-3m+2=0$$

or $$(m-2)(m-1)=0$$

i. e. $$m=1, 2.$$

$\therefore$ the solution is

$$z=f_1(y+x)+f_2(y+2x).$$

2. $$\frac{\partial^4 z}{\partial x^4}-2\frac{\partial^4 z}{\partial x^3\,\partial y}+2\frac{\partial^4 z}{\partial x\,\partial y^3}-\frac{\partial^4 z}{\partial y^4}=0.$$

The auxiliary equation is

$$m^4-2m^3+2m-1=0$$

or $$(m^2+1)(m^2-1)-2m(m^2-1)=0$$

or $$(m-1)(m+1)(m^2+1-2m)=0$$

or $$(m-1)^3(m+1)=0.$$

$$\therefore \quad m=1, 1, 1, -1.$$

$\therefore$ the solution is

$$z=\phi_1(y+x)+x\phi_2(y+x)+x^2\phi_3(y+x)+\phi_4(y-x).$$

Exercises 1 (1)

Solve—

1. $\frac{\partial^4 z}{\partial x^4}-\frac{\partial^4 z}{\partial y^4}=0.$

Ans. $z=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix).$

2. $(D^3-6D^2D'+11DD'^2-6D'^3)\,z=0.$

Ans. $z=\phi_1(y+x)+\phi_2(y+2x)+\phi_3(y+3x).$

3. $(D^3-4D^2D'+4DD'^2)\,z=0.$

Ans. $z=\phi_1(y+2x)+x\phi_2(y+2x)+\phi_3(y).$

4. $(D^3-3D^2D'+3DD'^2-D'^3)\,z=0.$

Ans. $z=\phi_1(y+x)+x\phi_2(y+x)+x^2\phi_3(y+x).$

1·2. The Particular Integral. Let us first take the case of a homogeneous equation.

Let the equation be

$$(D^n+A_1D^{n-1}D'+A_2D^{n-2}D'^2\ldots+A_nD'^n)\,z=V$$

or

$$(D-\alpha_1D')(D-\alpha_2D')\ldots(D-\alpha_nD')\,z=V.$$

Now symbolically, the particular integral

$$=\frac{1}{(D-\alpha_1D')(D-\alpha_2D')\ldots(D-\alpha_nD')}.V$$

$$=\frac{1}{D'^n}\,\frac{1}{\left(\frac{D}{D'}-\alpha_1\right)\left(\frac{D}{D'}-\alpha_2\right)\left(\frac{D}{D'}-\alpha_n\right)}.V$$

$$=\frac{1}{D'^n}\left\{\frac{A_1}{\frac{D}{D'}-\alpha_1}+\frac{A_2}{\frac{D}{D'}-\alpha_2}\cdots+\frac{A_n}{\frac{D}{D'}-\alpha_n}\right\}.V \quad \ldots(1)$$

(breaking into partial fractions).

Now let

$$\frac{1}{(D-\alpha_rD')}.V=\theta$$

or

$$(D-\alpha_rD')\,\theta=V=\psi(x,y).$$

$$\therefore\quad \theta e^{-x\alpha_rD'}=\int e^{-x\alpha_rD'}\,\psi(x,y)\,dx$$

$$\therefore \quad \theta = e^{x\alpha_r D'} \int \psi(x, y)\, e^{-x\alpha_r \cdot D'}\, dx$$

$$= e^{x \cdot \alpha_r D'} \int^x \psi(\xi, y)\, e^{-\alpha_r \xi \cdot D'}\, d\xi$$

$$= e^{x \cdot \alpha_r D'} \int^x \left(1 - \alpha_r \xi D' + \frac{\alpha_r^2 \xi^2 D'^2}{2!} - \ldots\right) \psi(\xi, y)\, d\xi$$

$$= e^{x\alpha_r D'} \int^x \left[\psi(\xi, y) - \alpha_r \xi \left(\frac{\partial}{\partial y}\right) \psi(\xi, y) + \frac{\alpha_r^2 \xi^2}{2!} \left(\frac{\partial}{\partial y}\right)^2 \psi(\xi, y) - \ldots\right] d\xi$$

$$= e^{x\alpha_r D'} \int^x \psi(\xi, y - \alpha_2 \xi)\, d\xi \quad \text{[by Taylor's theorem]}$$

$$= e^{\alpha_r x D'} \int^x \psi(\xi, y - \alpha_r \xi)\, d\xi$$

$$= \int_0^x \psi(\xi, y - \alpha_r \xi + \alpha_r x)\, d\xi \quad \text{[by Taylor's theorem]}$$

$$\text{P. I.} = \frac{1}{D'^{n-1}} \left\{ \sum_{r=1}^{n} \frac{A_r}{D - \alpha_r D} \right\} \psi(x, y) \qquad \text{[from (1)]}$$

$$= \frac{1}{D'^{n-1}} \sum_{r=1}^{n} A_r \int^x \psi(\xi, y - \alpha_r \xi + \alpha_r x)\, d\xi$$

$$= \iiint \ldots dy^{n-1} \left\{ \sum_{r=1}^{n} A_r \int^x \psi(\xi, y - \alpha_r \xi + \alpha_r x)\, d\xi \right\}.$$

Working Rule. To evaluate

$$\frac{1}{D - mD'} \phi(x, y),$$

form the function $\phi(x, y - mx)$, integrate it with respect to x, and in the integral obtained, change y into $y + mx$.

Note. This is the most general form of the solution. However, methods shorter than this can be employed in certain cases. We give them below.

1·3. The particular integral may be written

$$\frac{1}{F(D, D')}\psi(x, y).$$

Treating $F(D, D')$ as symbolic function of D and D', factorising it and resolving into partial fractions or expanding in an infinite series, the P. I. can be obtained.

For example,

$$\frac{1}{D^2-6DD'+9D'^2}(12x^2+36xy)$$

$$=\frac{1}{D^2}\frac{1}{\left(1-6\frac{D'}{D}+9\frac{D'^2}{D^2}\right)}(12x^2+36xy)$$

$$=\frac{1}{D^2}\left(1-\frac{3D'}{D}\right)^{-2}(12x^2+36xy)$$

$$=\frac{1}{D^2}\left[1+\frac{6D'}{D}+27\frac{D'^2}{D^2}+\ldots\right)(12x^2+36xy)\Big]$$

$$=\frac{1}{D^2}\left[(12x^2+36xy)+\frac{6}{D}(36.x)\right]$$

$$=x^4+6x^3y+6\times 36\times\frac{x^4}{2\times 3\times 4}$$

$$=10x^4+6x^3y.$$

1·31. When $\psi(x, y)$ is a function of $ax+by$, say $\phi(ax+by)$, then

$$\frac{1}{F(D, D')}\phi(ax+by)=\frac{1}{D^n F\left(\frac{D'}{D}\right)}\phi(ax+by).$$

Now $\frac{D'}{D}\phi(ax+by)=\frac{1}{D}\phi'(ax+by).b$

$$=\frac{b}{a}\phi(ax+by)$$

$$\therefore \quad \frac{1}{D^n F\left(\frac{D'}{D}\right)} \phi\,(ax+by) = \frac{1}{D^n F\left(\frac{b}{a}\right)} \phi\,(ax+by)$$

$$= \frac{1}{F\left(\frac{b}{a}\right)} \iiint \ldots \phi\,(ax+by)\,dx^n.$$

Note. If $\frac{b}{a}$ is a root of $F\left(\frac{D'}{D}\right)=0$.

$$\frac{1}{F\,(D, D')} \phi\,(ax+by) = \frac{1}{D^n\left(\frac{D'}{D}-\frac{b}{a}\right)\psi\left(\frac{D'}{D}\right)} \phi\,(ax+by)$$

$$= \frac{1}{D\left(\frac{D'}{D}-\frac{b}{a}\right)\psi\left(\frac{b}{a}\right)} \iiint \ldots \phi\,(ax+by)\,dx^{n-1},$$

which can be evaluated by the general rule.

1·32. A shorter method. When

$$f\,(x, y) = \phi\,(ax+by).$$

Then
$$D\phi\,(ax+by) = a\phi'\,(ax+by),$$
$$D^2\phi\,(ax+by) = a^2\phi''\,(ax+by),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$D^n\phi\,(ax+by) = a^n\phi^{(n)}\,(ax+by).$$

Also
$$D'\,(ax+by) = b\phi'\,(ax+by),$$
$$D'^2\,(ax+by) = b^2\phi''\,(ax+by),$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$D'^n\,(ax+by) = b^n\phi^{(n)}\,(ax+by).$$

Thus, if $F\,(D, D')$ is a homogeneous function of D and D' of degree n,

$$F\,(D, D')\,\phi\,(ax+by) = F\,(a, b)\,\phi^{(n)}\,(ax+by).$$

$$\therefore \quad \frac{\phi^{(n)}\,(ax+by)}{F\,(D, D')} = \frac{\phi\,(ax+by)}{F\,(a, b)}. \qquad \ldots(1)$$

provided $F(a, b) \neq 0$.

Put $ax+by=t$ in (1), then,

$$\frac{\phi^n(t)}{F(D, D')}=\frac{\phi(t)}{F(a, b)}.$$

Integrating with respect to t, n times,

$$\frac{\phi(t)}{F(D, D')}=\frac{\iint \ldots \int \phi(t)\,(dt)^n}{F(a, b)}$$

$$=\frac{\psi(t)}{F(a, b)},$$

where $\psi(t)$ is such that its nth differential coefficient with respect to t is $\phi(t)$.

Working Rule. In evaluating $\dfrac{\phi(ax+by)}{F(D, D')}$, where $F(D, D')$ is a rational integral homogeneous function of degree n, we integrate $\phi(ax+by)$ n times with respect to $ax+by$ considered as one variable and then divide the result by $F(a, b)$.

1·33. Exceptional case, when F (a, b)=0.

I. Consider the case

$$(bD-aD')\,z=x^r\phi(ax+by)$$

or

$$bp-aq=x^r\phi(ax+by), \qquad \ldots(1)$$

where

$$\frac{\partial z}{\partial x}=p,\ \frac{\partial z}{\partial y}=q.$$

Applying Lagrange's method to (1),

$$\frac{dx}{b}=\frac{dy}{-a}=\frac{dz}{x^r\phi(ax+by)}.$$

One solution is $\quad ax+by=c.$

The other solution is given by

$$\frac{dx}{b}=\frac{dz}{x^r\phi(c)}; \quad \therefore \quad z=\frac{x^{r+1}}{(r+1).b}\phi(c)$$

or

$$z=\frac{x^{r+1}}{b(r+1)}\phi(ax+by).$$

This is the solution of the given differential equation (1).

Thus $$\frac{1}{(bD-aD')}x^r\phi(ax+by)$$
$$=\frac{x^{r+1}}{b(r+1)}.\phi(ax+by). \qquad \ldots(F)$$

II. Next, consider
$$z=\frac{1}{(bD-aD')^n}.\phi(ax+by).$$

$$\frac{1}{(bD-aD')^n}\phi(ax+by)$$
$$=\frac{1}{(bD-aD')^{n-1}}\cdot\frac{1}{(bD-aD')}\phi(ax+by)$$
$$=\frac{1}{(bD-aD')^{n-1}}\cdot\frac{x}{b}\phi(ax+by) \qquad \text{[by (F) of I]}$$
$$=\frac{1}{(bD-aD')^{n-2}}\cdot\frac{1}{(bD-aD')}\cdot\frac{x}{b}\phi(ax+by)$$
$$=\frac{1}{(bD-aD')^{n-2}}\cdot\frac{1}{b}\cdot\frac{x^2}{2b}\phi(ax+by) \qquad \text{[by (F) of I]}$$
$$=\frac{1}{(bD-aD')^{n-2}}\cdot\frac{1}{b^2.2!}.x^2\phi(ax+by)$$
$$=\frac{1}{b^2.2!}\cdot\frac{1}{(bD-aD')^{n-2}}x^2\phi(ax+by)$$
$$=\frac{1}{b^2.2!}\cdot\frac{1}{(bD-aD')^{n-3}}\cdot\frac{1}{(bD-aD')}x^2\phi(ax+by)$$
$$=\frac{1}{b^2.2!}\cdot\frac{1}{(bD-aD')^{n-3}}\cdot\frac{x^3}{3b}\phi(ax+by) \qquad \text{[by (F) of I]}$$

... ...

... ...

$$=\frac{1}{b^{n-1}(n-1)!}\frac{1}{(bD-aD')^{n-n}}\cdot\frac{x^n}{nb}\phi(ax+by)$$
[repeatedly by (F) of I]

$$=\frac{x^n}{b^n.n!}\phi(ax+by).$$

Thus $\dfrac{\phi\,(\mathbf{ax+by})}{(\mathbf{bD-aD'})^n}=\dfrac{\mathbf{x}^n}{\mathbf{b}^n.\mathbf{n}\,!}\,\phi\,(\mathbf{ax+by}).$ ...(F)

Examples 1 (2)

1. *Solve* $\dfrac{\partial^2 z}{\partial x^2}+3\,\dfrac{\partial^2 z}{\partial x\,\partial y}+2\,\dfrac{\partial^2 z}{\partial y^2}=x+y.$ **(Agra 1952)**

The auxiliary equation is

$$m^2+3m+2=0;$$

$$\therefore\quad m=-2,\quad\text{or}\quad m=-1;$$

$$\therefore\quad \text{C. F.}=\phi_1\,(y-2x)+\phi_2\,(y-x).$$

Now $\text{P. I.}=\dfrac{1}{D^2+3DD'+2D'^2}\,(x+y)$

$$=\frac{1}{D^2\left(1+\dfrac{3D'}{D}+\dfrac{2D'^2}{D^2}\right)}\,x+\frac{1}{2D'^2\left(1+\dfrac{3D}{2D'}+\dfrac{D^2}{2D'^2}\right)}\,y$$

$$=\frac{1}{D^2}\,(x)+\frac{1}{2D'^2}\,y=\frac{x^3}{6}+\frac{y^3}{12}.$$

$\therefore$ the solution is

$$z=\phi_1\,(y-2x)+\phi_2\,(y-x)+\frac{x^3}{6}+\frac{y^3}{12}.$$

2. *Solve* $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\,\partial y}-6\,\dfrac{\partial^2 z}{\partial y^2}=xy.$

The auxiliary equation is

$$m^2-m-6=0.$$

$$\therefore\quad m=3\quad\text{or}\quad m=-2.$$

$$\therefore\quad \text{C.F.}=\phi_1\,(y+3x)+\phi_2\,(y-2x).$$

$$\text{P. I.}=\frac{1}{D^2-DD'-6D'^2}\,(xy)$$

$$=\frac{1}{D^2\left(1-\dfrac{D'}{D}-\dfrac{6D'^2}{D^2}\right)}\,(xy)$$

$$=\frac{1}{D^2}\left[\left(1+\frac{D'}{D}+\frac{6D'^2}{D}\right)+\ldots.\right]xy$$

$$=\frac{1}{D^2}\left(xy+\frac{x^2}{2}\right)$$

$$=\frac{x^3y}{6}+\frac{x^4}{24}.$$

$\therefore$ the solution is

$$z=\phi_1(y+3x)+\phi_2(y-2x)+\frac{x^3y}{6}+\frac{x^4}{24}.$$

3. *Solve* $\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=\cos mx \cos ny.$

The auxiliary equation is

$$m^2+1=0.$$

$$\therefore \quad m=\pm i.$$

$$\text{C. F.}=\phi_1(y+ix)+\phi_2(y-ix).$$

$$\text{P. F.}=\frac{\cos mx \cos ny}{D^2+D'^2}$$

$$=\frac{\cos mx \cos ny}{-m^2-n^2}=\frac{\cos mx \cos ny}{-(m^2+n^2)}.$$

$\therefore$ the solution is

$$z=\phi_1(y+ix)+\phi_2(y-ix)-\frac{\cos mx \cos ny}{m^2+n^2}.$$

4. *Solve* $\dfrac{\partial^3 z}{\partial x^2}-\dfrac{\partial^3 z}{\partial y^3}=x^3y^3.$

The auxiliary equation is

$$m^3-1=0$$

or

$$(m-1)(m^2+m+1)=0.$$

$$\therefore \quad m=1,\ m=(-1\pm\sqrt{-3})/2.$$

$$\text{C.F.}=\phi_1(y+x)+\phi_2\left(y+\frac{\sqrt{3}i-1}{2}x\right)+\phi_3\left(y+\frac{-\sqrt{3}i-1}{2}x\right).$$

$$\text{P.I.}=\frac{1}{D^3-D'^3}x^3y^3$$

$$=\frac{1}{D^3\left(1-\dfrac{D'^3}{D^3}\right)}x^3y^3$$

$$=\frac{1}{D^3}\left(1+\frac{D'^3}{D^3}+\ldots\right)x^3y^3$$

$$=\frac{1}{D^3}\left[(x^3y^3)+\frac{6.x^6}{4\times5\times6}\right]$$

$$=\frac{y^3.x^6}{4.5.6}+\frac{x^9}{4\times5\times7\times8\times9}.$$

$\therefore$ the solution is

$$z=\phi_1(y+x)+\phi_2\left(y+\frac{\sqrt{3}i-1}{2}x\right)+\phi_3\left(y+\frac{-\sqrt{3}i-1}{2}x\right)$$
$$+\frac{x^6y^3}{4.5.6}+\frac{x^9}{4.5.7.8.9}.$$

5. *Solve* $\dfrac{\partial^2 z}{\partial x^2}-2a\dfrac{\partial^2 z}{\partial x\,\partial y}+a^2\dfrac{\partial^2 z}{\partial y^2}=f(y+ax).$

The auxiliary equation is

$$m^2-2am+a^2=0$$

or

$$(m-a)^2=0.$$

$\therefore$ C.F.$=\phi_1(y+ax)+x.\phi_2(y+ax).$

$$\text{P.I.}=\frac{1}{(D-aD')^2}f(y+ax)$$

$$=\frac{1}{(D-aD')}\cdot\frac{1}{(D-aD')}f(y+ax).$$

Now let $\dfrac{1}{(D-aD')}f(y+ax)=u.$

$$\therefore\quad (D-aD')u=f(y+ax)$$

or

$$u.e^{-aD'x}=\int e^{-aD'x}f(y+ax)\,dx$$

$$=\int^x e^{-aD'\xi}f(y+a\xi)\,d\xi$$

$$=\int^x f(y)\,d\xi=xf(y).$$

$$\therefore\quad u=e^{aD'x}xf(y)$$

$$=xf(y+ax).$$

$$\therefore \quad \text{P.I.}=\frac{1}{(D-aD')}\, xf(y+ax)=v \text{ (say).}$$

$$\therefore \quad (D-aD')v=xf(y+ax).$$

$$\therefore \quad v.e^{-aD'x}=\int xe^{-aD'x} f(y+ax)\, dx$$

$$=\int^x \xi e^{-aD'\xi} f(y+a\xi)\, d\xi$$

$$=\int^x \xi f(y)\, d\xi$$

$$=\frac{x^2}{2} f(y).$$

$$\therefore \quad v=e^{aD'x} \cdot \frac{x^2}{2} f(y)$$

$$=\frac{x^2}{2} f(y+ax).$$

This follows at once from (F) of II of Art. 1·33.

6. *Solve*

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 u}{\partial z^3}-\frac{3\partial^3 u}{\partial x\, \partial y\, \partial z}=x^3-3xyz. \qquad \text{(Agra 1955)}$$

$$(D_1^3+D_2^3+D_3^3-3D_1D_2D_3)\, u=x^3-3xyz$$

or $$(D_1+D_2+D_3)\,(D_1^2+D_2^2+D_3^2-D_1D_2-D_2D_3-D_3D_1)\, u =x^3-3xyz$$

or $$(D_1+D_2+D_3)\,(D_1+\omega D_2+\omega^2 D_3)\,(D_1+\omega^2 D_2+\omega D_3) =x^3-3xyz$$

where ω is cube root of unity.

Now considering

$$(D_1+\omega D_2+\omega^2 D_3)\, u=0,$$

$$u=\phi(y, z)\, e^{(-\omega D_2-\omega^2 D_3)x}$$

ϕ being arbitrary,

or $$u=\phi(y-\omega x, z-\omega^2 x).$$

Therefore the complementary function is

$$\phi_1(y-x, z-x)+\phi_2(y-\omega x, z-\omega^2 x)+\phi_3(y-\omega^2 x, z-\omega x).$$

$$\text{Now P. I.}=\frac{1}{D_1^3+D_2^3+D_3^3-3D_1D_2D_3}(x^3-3xyz)$$

$$=\frac{x^3}{D_1^3\left(1+\frac{D_2^3}{D_1^3}+\frac{D_3^3}{D_1^3}-\frac{3D_2D_3}{D_1^2}\right)}$$

$$-\frac{3xyz}{-3D_1D_2D_3\left(-\frac{D_1^2}{3D_2D_3}-\frac{D_2^2}{3D_1D_3}-\frac{D_3^2}{3D_1D_2}+1\right)}$$

$$=\frac{x^3}{D_1^3}+\frac{xyz}{D_1D_2D_3}$$

$$=\frac{x^6}{4.5.6}+\frac{x^2.y^2.z^2}{2.2.2}.$$

$\therefore$ the solution is

$$u=\phi_1(y-x,\ z-x)+\phi_2(y-\omega x,\ z-\omega^2x)$$
$$+\phi_3(y-\omega^2x,\ z-\omega x)+\frac{x^6}{120}+\frac{x^2y^2z^2}{8}.$$

7. $(4D^2-4DD'+D'^2)z=16\log(x+2y)$ **(1960)**

or $$(2D-D')^2z=16\log(x+2y).$$

$\therefore$ the complementary function is

$$\phi_1(y+\tfrac{1}{2}x)+x\phi_2\left(y+\frac{x}{2}\right).$$

$$\text{Now}\quad \text{P. I.}=\frac{16\log(x+2y)}{(2D-D')^2}$$

$$=\frac{16}{4}\frac{\log(x+2y)}{\left(D-\frac{D'}{2}\right)\left(D-\frac{D'}{2}\right)}.$$

$$\text{Now let}\quad \frac{\log(x+2y)}{\left(D-\frac{D'}{2}\right)}=u.$$

$$\therefore\quad \left(D-\frac{D'}{2}\right)u=\log(x+2y);$$

$$\therefore\quad u=x\log(2y+x)\quad \text{(by general rule)}$$

$$\therefore \quad \text{P. I.} = \frac{4}{\left(D - \frac{D'}{2}\right)}\, x \log (2y + x)$$

$$= 2x^2 \log (x + 2y).$$

This follows at once from (F) of II of Art. 1·33.

$\therefore$ the solution is

$$z = 2x^2 \log (x + 2y) + f_1 (x + 2y) + x f_2 (x + 2y).$$

8. $(2D^2 - DD' - 3D'^2)\, z = 5e^{x-y}$

or $$(2D - 3D')\,(D + D') = 5e^{x-y}.$$

$\therefore$ the complementary function is

$$\phi_1 (y - x) + \phi_2 (2y + 3x).$$

$$\text{P. I.} = \frac{1}{(D + D')\,(2D - 3D')} \cdot 5e^{x-y}$$

$$= \frac{1}{(D + D')} \times \tfrac{1}{5} \cdot 5e^{x-y}, \quad [\text{by § } 1 \cdot 31]$$

$= xe^{x-y}$ (by general rule) or by II of 1·33.

9. $(D - aD')^2 = \phi (x) + \psi (y) + K (x + by).$

Clearly the C. F. is

$$\phi_1 (y + ax) + x\phi_2 (y + ax).$$

$$\text{P.I.} = \frac{\phi (x) + \psi (y) + K (x + by)}{(D - aD')^2}.$$

Now $$\frac{\phi (x)}{(D - aD')^2} = \frac{\phi (x)}{D^2 \left(1 - \frac{aD'}{D}\right)^2} = \frac{\phi (x)}{D^2}$$

$$= \iint \phi (x)\, dx^2.$$

$$\frac{\psi (y)}{(D - aD'^2)} = \frac{\psi (y)}{a^2 D'^2 \left(1 - \frac{D}{aD'}\right)^2}$$

$$= \frac{\iint \psi (y)\, dy^2}{a^2}$$

Now, $$\frac{1}{(D - aD')^2}\, K (x + by)$$

$$=\frac{1}{D^2\left(1-\frac{aD'}{D}\right)^2} K(x+by)$$

$$=\frac{1}{D^2(1-ab)^2} K(x+by)$$

$$=\frac{1}{(1-ab)^2}\iint K(x+by)\,dx^2.$$

$$\therefore \quad \frac{K(x+by)}{(D-aD')^2}$$

$$=\frac{1}{(1-ab)^2}\iint K(t)\,dt^2,$$

putting $t=x+by$ after integration.

Hence the solution is

$$z=\phi_1(y+ax)+x\phi_2(y+ax)+\iint\phi(x)\,dx^2+(1/2a^2)\iint\psi(y)\,dy^2$$
$$+\frac{1}{(1-ab)^2}\iint K(t)\,dt^2.$$

where $t=x+by$, after integration.

Exercises 1 (2)

1. $\dfrac{\partial^2 u}{\partial x^2}-\dfrac{a^2\partial^2 u}{\partial y^2}=x^2.$

Ans. $u=\phi_1(y+ax)+\phi_2(y-ax)+\dfrac{x^4}{12}.$

2. (a) $\dfrac{\partial z}{\partial x}+\dfrac{\partial z}{\partial y}=\sin x.$ Ans. $z=\phi_1(y-x)-\cos x.$

(b) $\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}=2\sin x\cos y.$

Ans. $z=\phi_1(y+x)+x\sin(x+y)-\frac{1}{2}\cos(x-y).$

3. $(D^2-2DD'+D'^2)\,z=e^{x+2y}.$

Ans. $z=e^{x+2y}+f(y+x)+xF(y+x).$

4. $(D^2-6DD'+9D'^2)\,z=6x+2y.$

Ans. $z=x^2(3x+y)+\phi_1(y+3x)+x\phi_2(y+3x).$

5. $(D^3-4D^2D'+4DD'^2)\,z=4\sin(2x+y).$

Ans. $z=-x^2\cos(2x+y)+\phi_1(y+2x)$
$+x\phi_2(y+2x)+\phi_3(y).$

6. $(D-2D')(D+D')z=(y-1)e^x$.

Ans. $z=ye^x+\phi_1(y+2x)+\phi_2(y-x)$.

7. $(D^2+2DD'+D'^2)z=2\cos y-x\sin y$.

Ans. $z=x\sin y+f_1(y-x)+xf_2(y-x)$.

8. $(D-D')^2 z=x+\phi(x+y)$.

Ans. $z=\dfrac{x^3}{6}+\dfrac{x^2}{2}\phi(y+x)+x\psi_1(x+y)+\psi_2(x+y)$.

9. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial z\,\partial x}-\dfrac{\partial^2 u}{\partial y^2}-\dfrac{\partial^2 y}{\partial y\,\partial z}=xyz$.

Ans. $u=\phi(y+x, z)+\psi(y-x, z-x)$
$+x^3yz/6-x^4y/24+x^5/120$.

1·4. The non-homogeneous equation with constant coefficients.

The simplest case is

$$(D-mD'-\alpha)z=0$$

or

$$z=e^{(mD'+\alpha)x}\phi(y),$$

where D' has been considered algebraic and ϕ is arbitrary

$$=e^{\alpha x}\phi(y+mx).$$

Note. Also $\quad (D-mD'-\alpha)z=0$

or

$$p-mq=\alpha z.$$

$\therefore$ the subsidiary equations are

$$\frac{dx}{1}=\frac{dy}{-m}=\frac{dz}{\alpha z}.$$

$$\therefore \quad z=e^{\alpha x}\phi(y+mx).$$

1·41. Similarly the integral of

$$(D-m_1D'-\alpha_1)(D-m_2D'-\alpha_2),$$
$$(D-m_3D'-\alpha_3)\ldots=0$$

is $z=e^{\alpha_1 x}\phi_1(y+m_1x)+e^{\alpha_2 x}\phi_2(y+m_2x)+e^{\alpha_3 x}\phi_3(y+m_3x)+\ldots$

1·42. In case of repeated factors,

$$(D-mD'-\alpha)^2 z=0 \qquad \ldots(1)$$

or

$$(D-mD'-\alpha)(D-mD'-\alpha)z=0.$$

Let $(D-mD-\alpha)\,z=v$.

Then, $$(D-mD'-\alpha)\,v=0 \qquad \text{[from (1)]}$$

or, $$v=e^{\alpha x}\,\phi_1\,(y+mx)$$

or, $$(D-mD'-\alpha)\,z=e^{\alpha x}\,\phi_1\,(y+mx)\,;$$

$$\therefore\quad z=e^{(mD'+\alpha)x}\left[\int\{e^{-(mD'+\alpha)x}\times e^{\alpha x}\phi_1\,(y+mx)\}\,dx+\phi_2\,(y)\right]$$

$$=e^{(\alpha x+mD')x}\int\phi_1\,(y)\,dx+e^{\alpha x}e^{mxD'}\phi_2(y)$$

$$=e^{\alpha x}\,x\phi_1\,(y+mx)+e^{\alpha x}\phi_2\,(y+mx).$$

Similarly proceeding in the case of $(D-mD'-\alpha)^r\,z=0$, we have

$$z=e^{\alpha x}\phi_1\,(y+mx)+e^{\alpha x}.x\phi_2\,(y+mx)+e^{\alpha x}x^2\phi_3\,(y+mx)+\dots$$
$$\dots+e^{\alpha x}x^{r-1}\phi_r\,(y+mx).$$

1·5. The Particular Integral.

The methods for obtaining particular integrals of non-homogeneous partial differential equations are very similar to those used in solving linear equations with constant coefficients. Here we shall consider only a few examples.

Note. It can be easily shown that

$$\frac{1}{F\,(D,\,D')}\,e^{ax+by}=\frac{e^{ax+by}}{F\,(a,\,b)},\qquad \text{if } F\,(a,\,b)\neq 0$$

and $$\frac{1}{F\,(D^2,\,DD',\,D'^2)}\,\sin\,(ax+by)$$

$$=\frac{1}{F\,(-a^2,\,-ab,\,-b^2)}\,\sin\,(ax+by),$$

provided the denominator is not zero.

Examples 1 (3)

1. *Solve* $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}-3\,\dfrac{\partial z}{\partial x}+3\,\dfrac{\partial z}{\partial y}=xy+e^{x+2y}$. **(Agra 1958)**

Here, $$(D^2-D'^2-3D+3D')\,z=xy+e^{x+2y}$$

or, $$(D-D')\,(D+D'-3)\,z=xy+e^{x+2y}.$$

$\therefore$ the complementary function is

$$\phi\,(x+y)+e^{3x}\psi\,(y-x).$$

Now $\quad$ $$\text{P. I.}=\frac{xy}{(D-D')(D+D'-3)}+\frac{e^{x+2y}}{(D-D')(D+D'-3)}$$

$$=\frac{1}{3(D'-D)\left[1-\frac{D'+D}{3}\right]}xy+\frac{e^{x+2y}}{(1-D')(1+D'-3)}$$

$$=\frac{1}{3(D'-D)}\left[1+\frac{(D'+D)}{3}+\frac{(D'+D)^2}{9}+\dots\right]xy+\frac{e^x.e^{2y}}{(-1)(D'-2)}$$

$$=-\frac{1}{3(D'-D)}\left[xy+\frac{x}{3}+\frac{y}{3}+\frac{2}{9}\right]-e^x.e^{2y}\times\frac{1}{D'}.1$$

$$=\frac{1}{-3D\left(1-\frac{D'}{D}\right)}\left(xy+\frac{x}{3}+\frac{y}{3}+\frac{2}{9}\right)-ye^{x+2y}$$

$$=-\frac{1}{3D}\left(1+\frac{D'}{D}+\frac{D'^2}{D^2}\dots\right)\left(xy+\frac{x}{3}+\frac{y}{3}+\frac{2}{9}\right)-ye^{x+2y}$$

$$=-\frac{1}{3D}\left[xy+\frac{x}{3}+\frac{y}{3}+\frac{2}{9}+\frac{x^2}{2}+\frac{x}{3}\right]-ye^{x+2y}$$

$$=-\frac{x^2y}{3.2}-\frac{1}{9}.\frac{x^2}{2}-\frac{xy}{9}-\frac{2x}{27}-\frac{x^3}{18}-\frac{x^2}{18}-ye^{x+2y}.$$

$\therefore$ the solution is

$$z=\phi(x+y)+e^{3x}\psi(y-x)-\frac{x^2y}{6}-\frac{x^2}{9}-\frac{xy}{9}-\frac{2x}{27}-\frac{x^3}{18}-ye^{x+2y}.$$

2. $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}+\dfrac{\partial z}{\partial x}+3\dfrac{\partial z}{\partial y}-2z=e^{x-y}-x^2y.$ **(Agra 1957)**

$$[(D-D')(D+D')+2(D+D')-(D-D')-2]\,z=e^{x-y}-x^2y$$

or, $$[(D-D'+2)(D+D'-1)]\,z=e^{x-y}-x^2y.$$

$\therefore$ the complementary function is

$$z=e^{-2x}\phi(y+x)+e^x\psi(y-x).$$

Now,

$$\text{P.I.}=\frac{e^{x-y}}{(D-D'+2)(D+D'-1)}-\frac{x^2y}{D^2-D'^2+D+3D'-2}$$

$$=\frac{e^{x-y}}{(1-D'+2)\,D'}-\frac{x^2y}{-2\left[1-\left\{\frac{D}{2}+\frac{3D'}{2}-\frac{D'^2}{2}+\frac{D^2}{2}\right\}\right]}$$

$$=\frac{e^{x-y}}{-4}+\tfrac{1}{2}\left[1+\left(\frac{D}{2}+\frac{3D'}{2}-\frac{D'^2}{2}+\frac{D^2}{2}\right)\right.$$

$$+\left(\frac{D}{2}+\frac{3D'}{2}-\frac{D'^2}{2}+\frac{D^2}{2}\right)^2$$

$$\left.+\left(\frac{D}{2}+\frac{3D'}{2}-\frac{D'^2}{2}+\frac{D^2}{2}\right)^3+\ldots\right]x^2y$$

$$==\frac{e^{x-y}}{4}+\tfrac{1}{2}\left[1+\frac{D}{2}+\frac{3D'}{2}+\frac{D^2}{2}+\frac{D^2}{4}+\frac{3DD'}{2}+\frac{3D^2D'}{2}\right.$$

$$\left.+\frac{3D^2D'}{4}+\frac{3D^2D'}{8}+\ldots\right]x^2y$$

$$=-\frac{e^{x-y}}{4}+\tfrac{1}{2}\left[x^2y+xy+\frac{3x^2}{2}+y+\frac{y}{2}+3x+3+\tfrac{3}{2}+\tfrac{3}{4}\right]$$

$$=-\frac{e^{x-y}}{4}+\frac{x^2y}{2}+\frac{3x^2}{4}+\frac{3y}{4}+\frac{xy}{2}+\frac{3x}{2}+\frac{21}{8}.$$

$\therefore$ the solution is

$$z=e^{-2x}\,\phi\,(y+x)+e^{x}\,\psi\,(y-x)-\frac{e^{x-y}}{4}+\frac{x^2y}{2}+\frac{3x^2}{4}+\frac{3y}{4}$$

$$+\frac{xy}{2}+\frac{3x}{2}+\frac{21}{8}.$$

3. $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial x\,\partial y}+\dfrac{\partial z}{\partial y}-z=\cos\,(x+2y)+e^{y}.$ **(Agra 1955)**

$$(D^2-DD'+D'-1)\,z=\cos\,(x+2y)+e^{y},$$

$$(D-1)\,(D-D'+1)\,z=\cos\,(x+2y)+e^{y}.$$

$\therefore$ the complementary function is

$$e^{x}.\phi\,(y)+e^{-x}\,\psi\,(y+x).$$

$$\text{P.I.}=\frac{1}{D^2-DD'+D'-1}\,[\cos\,(x+2y)+e^{y}]$$

$$=\frac{1}{-1+2+D'-1}\cos\,(x+2y)+\frac{e^{y}}{D^2-D+1-1}$$

$$=\frac{1}{D'}\cos(x+2y)+e^{y}\frac{1}{-D(1-D)}$$

$$=\frac{\sin(x+2y)}{2}-xe^{y}.$$

$\therefore$ the solution is

$$z=e^{x}\phi(y)+e^{-x}\psi(y+x)+\tfrac{1}{2}\sin(x+2y)-xe^{y}.$$

4. *Solve*

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}-2\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial y}+2\frac{\partial z}{\partial x}$$
$$=e^{2x+3y}+\sin(2x+y)+xy.$$

(Agra 1963, 56, 54)

$$(D^2-DD'-2D'^2+2D'+2D)z=e^{2x+3y}+\sin(2x+y)+xy$$

or $$(D+D')(D-2D'+2)z=e^{2x+2y}+\sin(2x+y)+xy.$$

Therefore the complementary function is

$$\phi(y-x)+e^{-2x}\psi(y+2x).$$

Now $$\frac{1}{D^2-DD'-2D'^2+2D'+2D}e^{2x+2y}$$

$$=\frac{1}{4-6-18+6+4}e^{2x+3y}=-\frac{e^{2x+3y}}{10}.$$

Also, $$\frac{1}{D^2-DD'-2D'^2+2D'+2D}\sin(2x+y)$$

$$=\frac{1}{-4+2+2+2D+2D'}\sin(2x+y)$$

$$=\frac{1}{2(D+D')}\sin(2x+y)=\frac{D-D'}{2(D^2-D'^2)}\sin(2x+y)$$

$$=\frac{(D-D')\sin(2x+y)}{-6}$$

$$=-\tfrac{1}{6}[2\cos(2x+y)-\cos(2x+y)]$$

$$=-\frac{\cos(2x+y)}{6}$$

and, $\dfrac{1}{D^2-DD'-2D'^2+2D+2D'}\,xy$

$$=\frac{1}{(D+D')\,2\left[1-\left(D'-\frac{D}{2}\right)\right]}\,xy$$

$$=\frac{1}{2\,(D+D')}\left[1+\left(D'-\frac{D}{2}\right)+\left(D'-\frac{D}{2}\right)^2+\ldots\right]xy$$

$$=\frac{1}{2\,(D+D')}\left(xy+x-\frac{y}{2}-1\right)$$

$$=\frac{1}{2D}\left(1-\frac{D'}{D}\right)\left(xy+x-\frac{y}{2}-1\right)$$

$$=\frac{1}{2D}\left(xy+x-\frac{y}{2}-1-\frac{x^2}{2}+\frac{x}{2}\right)$$

$$=\tfrac{1}{2}\left(\frac{x^2y}{2}+\frac{3x^2}{4}-\frac{x^3}{6}-\frac{xy}{2}-x\right)$$

$$=\frac{1}{24}\,(6x^2y+9x^2-2x^3-6xy-12x).$$

$\therefore$ the solution is

$$z=\phi\,(y-x)+e^{-2x}\,\psi\,(y+2x)-\frac{e^{2x+3y}}{10}-\frac{\cos\,(2x+y)}{6}$$
$$+\frac{x}{24}\,(6xy+9x-2x^2-6y-12).$$

5. $$mn\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)-(m^2+n^2)\,\frac{\partial^2 z}{\partial x\,\partial y}+mn\left(n\,\frac{\partial z}{\partial x}-m\,\frac{\partial z}{\partial y}\right)$$
$$=\cos\,(mx+ny)+\cos\,(kx+ly)$$

or $$(mnD^2+mnD'^2-m^2DD'-n^2DD'+mn^2D-m^2nD')z$$
$$=\cos\,(mx+ny)+\cos\,(kx+ly)$$

or $$[mD\,(nD-mD')-nD'\,(nD-mD')+mn\,(nD-mD')]z$$
$$=\cos\,(mx+ny)+\cos\,(kx+ly)$$

or $$[(nD-mD')\,(mD-nD'+mn)]z$$
$$=\cos\,(mx+ny)+\cos\,(kx+ly).$$

$\therefore$ the complementary function is

$$\phi_1\left(y+\frac{m}{n}x\right)+e^{-mnx}\,\phi_2\left(y+\frac{n}{m}x\right).$$

Now $\dfrac{\cos(kx+ly)}{mnD^2+mnD'^2-(m^2+n^2)\,DD'-m^2nD'+mn^2D}$

$$=\frac{\cos(kx+ly)}{-mnk^2-mnl^2+(m^2+n^2)\,kl-m^2nD'+mn^2D}$$

$$=\frac{\cos(kx+ly)}{[(mn^2D-m^2nD')+t]},$$

where $\quad t=-mn\,(k^2+l^2)+(m^2+n^2)\,kl$

$$=\frac{[mn\,(nD-mD')-t]\cos(kx+ly)}{[+m^2n^2\,(nD-mD')^2-t^2]}$$

$$=\frac{\{-mn\,(nD-mD')+t\}\cos(kx+ly)}{-m^2n^2\,\{-n^2k^2-m^2l^2+2mnk\,l\}+t^2}$$

$$=mn^2k\sin(kx+ly)-m^2nl\sin(kx+ly)$$
$$+(nk-ml)\,(nl-mk)\cos(kx+ly)$$
$$\div m^2n^2\,[nk-ml]^2+[-mk\,(nk-ml)+nl\,(kn-ml)]^2$$

$$=mn\sin(kx+ly)\,[nk-ml]$$
$$+(nk-ml)\,(nl-mk)\cos(kx+ly)$$
$$\div(nk-ml)^2\,[(nl-mk)^2+m^2n^2]$$

$$=\frac{mn\sin(kn+ly)-(mk-nl)\cos(kx+ly)}{(nk-ml)\,[(nl-mk)^2+m^2n^2]}$$

and $\quad \dfrac{\cos(mx+ny)}{(nmD^2+mnD'^2)-(m^2+n^2)\,DD'+mn^2D-m^2nD'}$

$$=\frac{\cos(mx+ny)}{-mn\,(n^2+m^2)+(m^2+n^2)\,mn+mn\,(nD-mD')}$$

$$=\frac{\cos(mx+ny)}{mn\,(nD-mD')}=\frac{\cos(mx+ny)}{mn^2\left(D-\dfrac{m}{n}D'\right)}$$

$$=\frac{x}{mn^2}\cos(mx+ny).$$

$\therefore$ the solution is

$$z=\phi_1\left(y+\frac{m}{n}x\right)+e^{-mnx}\phi_2\left(y+\frac{n}{m}x\right)$$

$$+\frac{x}{mn^2}\cos(mx+ny)$$

$$+\frac{mn\sin(kx+ly)-(mk-nl)\cos(kx+ly)}{(nk-ml)\,[(nl-mk)^2+m^2n^2]}.$$

Caution. It may be noticed that by a different method of procedure such as expanding in powers of D/D' or D'/D, the particular integrals obtained may be different in form but they can be transformed into each other with the help of the complementary function.

Exercises 1 (3)

1. $$\frac{\partial^3 u}{\partial x^2\,\partial y}-2\frac{\partial^3 u}{\partial x\,\partial y^2}-3\frac{\partial^3 u}{\partial x^2\,\partial z}-3\frac{\partial^3 u}{\partial x\,\partial z^2}-2\frac{\partial^3 u}{\partial y^2\,\partial z}+6\frac{\partial^3 u}{\partial y\,\partial z^2}+7\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=0.$$

   Ans. $u=\phi_1(z-x,\ y)+\phi_2(y+2x,\ z)+\phi_3(z+3y,\ x)$.

2. $$\frac{\partial^2 z}{\partial x^2}-a^2\frac{\partial^2 z}{\partial y^2}+2ab\frac{\partial z}{\partial x}+2a^2b\frac{\partial z}{\partial y}=0.$$

   Ans. $z=\phi_1(y-ax)+e^{-abx}\phi_2(y+ax)$.

3. $(D^2+D'^2-2DD'-3D+3D'+2)\,z=e^{2x-y}$.

   Ans. $z=e^{x}\phi_1(y+x)+e^{2x}\phi_2(y+x)+\dfrac{e^{2x-y}}{2}$.

4. $(D-3D'-2)^2\,z=2e^{2x}\tan(y+3x)$.

   Ans. $z=e^{2x}\,[x^2\tan(y+3x)+xf_1(y+3x)+f_2(y+3x)]$.

5. $$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 u}{\partial z^3}-3\frac{\partial^3 u}{\partial x\,\partial y\,\partial z}=x^3+y^3+z^3-3xyz.$$

   Ans. $$u=\phi_1(y-x,\ z-x)+\phi_2(y-\omega x,\ z-\omega^2 x)+\phi_3(y-\omega^2 x,\ z-\omega x)+\frac{x^6+y^6+z^6}{4.5.6}+\frac{x^2y^2z^2}{8}.$$

6. $(D+D'-1)(D+2D'-3)\,z=4+3x+6y$.

Ans. $z=6+x+2y+e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-2x)$.

1·6. Equations reducible to Homogeneous Linear Form.

An equation in which the coefficient of a differential coefficient of any order is a constant multiple of the variables of the same degree, may be transformed into one having constant coefficients.

Examples 1 (4)

1. *Solve* $x^2\dfrac{\partial^2 z}{\partial x^2}-y^2\dfrac{\partial^2 z}{\partial y^2}-y\dfrac{\partial z}{\partial y}+x\dfrac{\partial z}{\partial x}=0.$ **(Agra 1953)**

Assume $u=\log x$, $v=\log y$. Then

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\times\frac{1}{x}.$$

or,
$$x\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u},\ \text{so that}\ x\frac{\partial}{\partial x}=\frac{\partial}{\partial u}. \qquad \ldots(1)$$

$$\therefore\quad x\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial x}\right)=x^2\frac{\partial^2 z}{\partial x^2}+x\frac{\partial z}{\partial x}=\frac{\partial^2 z}{\partial u^2}\ \text{[from (1)]}.$$

Similarly,
$$y^2\frac{\partial^2 z}{\partial y^2}+y\frac{\partial z}{\partial y}=\frac{\partial^2 z}{\partial v^2}.$$

$\therefore$ the given equation reduces to

$$\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}=0,$$

for which

$$\begin{aligned}z&=\phi(u+v)+\psi(v-u)\\&=\phi[\log x+\log y]+\psi[\log y-\log x]\\&=\phi(\log xy)+\psi\left[\log\left(\frac{y}{x}\right)\right]\\&=f_1(xy)+f_2\left(\frac{y}{x}\right).\end{aligned}$$

2. *Solve* $x^2 \frac{\partial^2 z}{\partial x^2} - 4xy \frac{\partial^2 z}{\partial x \, \partial y} + 4y^2 \frac{\partial^2 z}{\partial y^2} + 6y \frac{\partial z}{\partial y} = x^3 y^4.$

(Agra 1957)

As shown in the last example, if $u = \log x$, $v = \log y$,

$$x \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}, \; y \frac{\partial z}{\partial y} = \frac{\partial z}{\partial v}.$$

$$x^2 \frac{\partial^2 z}{\partial x^2} + x \frac{\partial z}{\partial x} = \frac{\partial^2 z}{\partial u^2} \text{ and } y^2 \frac{\partial^2 z}{\partial y^2} + y \frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial v^2}.$$

Now
$$y \frac{\partial}{\partial y}\left(x \frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)$$

or
$$yx \frac{\partial^2 z}{\partial x \, \partial y} = \frac{\partial^2 z}{\partial v \, \partial u}.$$

With these substitutions the equation takes the form

$$\frac{\partial^2 z}{\partial u^2} - \frac{\partial z}{\partial u} - 4 \frac{\partial^2 z}{\partial u \, \partial v} + 4 \frac{\partial^2 z}{\partial v^2} - 4 \frac{\partial z}{\partial v} + 6 \frac{\partial z}{\partial v} = e^{3u} . e^{4v}$$

or
$$\frac{\partial^2 z}{\partial u^2} - 4 \frac{\partial^2 z}{\partial u \, \partial v} + 4 \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} + 2 \frac{\partial z}{\partial v} = e^{3u+4v}. \qquad \ldots(1)$$

Denoting $\frac{\partial}{\partial u}$ by D and $\frac{\partial}{\partial v}$ by D' in (1),

$$(D^2 - 4DD' + 4D^2 - D + 2D') \, z = e^{3u+4v}$$

or
$$[(D - 2D')\,(D - 2D' - 1)] \, z = e^{3u+4v}.$$

$\therefore$ the complementary function is

$$\phi_1 \, (v + 2u) + e^u \phi_2 \, (v + 2u),$$
$$= \phi_1 \, (\log x^2 y) + x \phi_2 \, (\log x^2 y)$$
$$= \phi \, (x^2 y) + x \psi \, (x^2 y).$$

$$\text{P. I.} = \frac{1}{(D - 2D')\,(D - 2D' - 1)} \, e^{3u+4v}$$

$$= \frac{1}{(-5)\,(-6)} \, e^{3u+4v}$$

$$= \frac{x^3 y^4}{30}.$$

$\therefore$ the solution is

$$z=\phi\ (x^2y)+x\psi\ (x^2y)+\frac{x^3y^4}{30}.$$

3. *Solve* $\frac{1}{x^2}\frac{\partial^2 z}{\partial x^2}-\frac{1}{x^3}\frac{\partial z}{\partial x}=\frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}-\frac{1}{y^3}\frac{\partial z}{\partial y}.$ **(Agra 1958)**

Assuming $x^2=u$ and $y^2=v$,

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\times 2x \quad \text{and} \quad \frac{\partial z}{\partial y}=2y\ \frac{\partial z}{\partial v},$$

$$\frac{1}{2x}\cdot\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u} \text{ and } \frac{1}{2y}\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}$$

so that
$$\frac{1}{2x}\frac{\partial}{\partial x}=\frac{\partial}{\partial u} \text{ etc.}$$

or
$$\frac{1}{2x}\frac{\partial}{\partial x}\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial u^2}$$

or
$$-\frac{1}{4x^3}\frac{\partial z}{\partial x}+\frac{1}{4x^2}\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial u^2}.$$

$\therefore$ the given equation reduces to

$$\frac{\partial^2 z}{\partial u^2}=\frac{\partial z^2}{\partial v^2}$$

or
$$\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}=0.$$

$\therefore$ the solution is

$$z=\phi_1\ (v+u)+\phi_2\ (v-u)$$
$$=\phi_1\ (y^2+x^2)+\phi_2\ (y^2-x^2)$$

4. *Solve* $x^2\frac{\partial^2 u}{\partial x^2}+2xy\ \frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(x^2+y^2)^{n/2}.$

Putting $x=e^p$, $y=e^q$, the equation reduces to

$$\frac{\partial^2 u}{\partial p^2}-\frac{\partial u}{\partial p}+2\ \frac{\partial^2 u}{\partial p\ \partial q}+\frac{\partial^2 u}{\partial q^2}-\frac{\partial u}{\partial q}=(e^{2p}+e^{2q})^{n/2}$$

if
$$\theta=\frac{\partial}{\partial p},\ \phi=\frac{\partial}{\partial q},$$

$$(\theta^2-\theta+2\theta\phi+\phi^2-\phi)\ u=(e^{2p}+e^{2q})^{n/2},$$

$$(\theta+\phi)\ (\theta+\phi-1)\ u=(e^{2p}+e^{2q})^{n/2}.$$

$\therefore$ the complementary function is

$$u=\phi_1\ (q-p)+e^p\phi_2\ (q-p)$$
$$=\phi_1\ (y/x)+x\phi_2\ (y/x).$$

$$\text{P. I.}=\frac{(e^{2p}+e^{2q})^{n/2}}{(\theta+\phi)\ (\theta+\phi-1)}$$

$$=\frac{e^{np}\ [1+e^{2(q-p)}]^{n/2}}{(\theta+\phi)\ (\theta+\phi-1)}$$

$$=\frac{e^{np}\left\{1+\frac{n}{2}e^{2(q-p)}+\frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{1.2}e^{4(q-p)}+\ldots\right\}}{(\theta+\phi)\ (\theta+\phi-1)}$$

$$=\frac{e^{np}+\frac{n}{2}e^{2q+(n-2)p}+\frac{\frac{n}{2}\left(\frac{n}{2}-1\right)}{1.2}e^{4q+(n-4)p}+\ldots}{(\theta+\phi)\ (\theta+\phi-1)}. \quad \ldots(1)$$

Now for $m=0,\ 1,\ 2, \ldots,$

$$\frac{e^{2mq+(n-2m)p}}{(\theta+\phi)\ (\theta+\phi-1)}=\frac{e^{[2mq+(n-2m)p]}}{n^2-n} \qquad \ldots(2)$$

$$\therefore\quad \text{P. I.}=\frac{e^{np}\left[1+\frac{n}{2}e^{2(q-p)}+\ldots\right]}{n^2-n}=\frac{x^n\left[1+\frac{y^2}{x^2}\right]^{n/2}}{n^2-n}$$

[from (1) and (2)]

$$=\frac{(x^2+y^2)^{n/2}}{n\ (n-1)}.$$

$\therefore$ the solution is

$$u=\phi_1\left(\frac{y}{x}\right)+n\phi_2\left(\frac{y}{x}\right)+\frac{(x^2+y^2)^{n/2}}{n\ (n-1)}$$

5. $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2 u+n^2u=0.$

Putting $x=e^{\xi}$, $y=e^{\eta}$, $z=e^{\zeta}$,

i.e. $x\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}=D_1$, $y\frac{\partial}{\partial y}+\frac{\partial}{\partial \eta}=D_2$, $z\frac{\partial}{\partial z}=\frac{\partial}{\partial \zeta}=D_3$,

the equation reduces to

$$\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}+\frac{\partial}{\partial \zeta}\right)^2 u+n^2u=0$$

$$[(D_1+D_2+D_3)^2+n^2]\, u=0$$

or
$$[D_1+D_2+D_3\pm in]\, u=0.$$

$$\therefore\quad u=e^{in\xi}\,\phi_1(\zeta-\xi,\ \eta-\xi)+e^{-in\xi}\,\phi_2(\zeta-\xi,\ \eta-\xi)$$

$$=(\cos n\xi+i\sin n\xi)\,\phi_1\left(\log\frac{z}{x},\ \log\frac{y}{x}\right)$$
$$+(\cos n\xi-i\sin n\xi)\,\phi_2\left(\log\frac{z}{x},\ \log\frac{y}{x}\right)$$

$$=F_1\left(\frac{z}{x},\ \frac{y}{x}\right)\cos(n\log x)$$
$$+F_2\left(\frac{z}{x},\ \frac{y}{x}\right)\sin(n\log x).$$

6. $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+nu$
$$=n\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)+x^2+y^2+x^3.$$

Assuming $x=e^p$, $y=e^q$, the given equation reduces to, if θ denotes $\frac{\partial}{\partial p}$ and ϕ, $\frac{\partial}{\partial q}$,

$$(\theta^2-\theta+2\theta\phi+\phi^2-\phi+n)u=n(\theta+\phi)+e^{2p}+e^{2q}+e^{3p}$$

or
$$[\theta^2-\theta+2\theta\phi+\phi^2-\phi+n-n\theta-n\phi]\,u=e^{2p}+e^{2p}+e^{3p}$$

or
$$[(\theta+\phi-1)(\theta+\phi-n)]\,u=e^{2p}+e^{2q}+e^{3p}.$$

$\therefore$ the complementary function is

$$u=e^p f_1(q-p)+e^{np}f_2(q-p)$$

$$=xF_1\left(\frac{y}{x}\right)+x^nF_2\left(\frac{y}{x}\right).$$

$$\text{P. I.}=\frac{e^{2p}}{(\theta+\phi-1)(\theta+\phi-n)}+\frac{e^{2q}}{(\theta+\phi-1)(\theta+\phi-n)}+\frac{e^{3p}}{(\theta+\phi-1)(\theta+\phi-n)}$$

$$=\frac{e^{2p}}{2-n}+\frac{e^{2q}}{2-n}+\frac{e^{3p}}{2(3-n)}.$$

$\therefore$ the solution is

$$u=xF_1\left(\frac{y}{x}\right)+x^nF_2\left(\frac{y}{x}\right)+\frac{x^2+y^2}{2-n}+\frac{x^3}{2(3-n)}.$$

Exercises 1 (4)

1. Solve $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=xy.$

   Ans. $z=\phi_1(xy)+x\phi_2\left(\frac{y}{x}\right)+xy\log x.$

2. $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=x^my^n.$

   Ans. $z=f\left(\frac{y}{x}\right)+xF\left(\frac{y}{x}\right)+\frac{x^my^n}{(m+n)(m+n-1)}.$

3. $x^2\frac{\partial^2 z}{\partial x^2}-3xy\frac{\partial^2 z}{\partial x\,\partial y}+2y^2\frac{\partial^2 z}{\partial y^2}+5y\frac{\partial z}{\partial y}-2z=0.$

   Ans. $z=x^2\phi_1(xy)+\frac{1}{x}\phi_2(x^2y)$

4. $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}-z=0.$

   Ans. $z=x\phi_1\left(\frac{y}{x}\right)+\frac{1}{x}\phi_2\left(\frac{y}{x}\right).$

5. $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}=x^2y$ **Ans.** $z=\phi_1(xy)+x\phi_2\left(\frac{y}{x}\right)+\frac{x^2y}{2}.$

1·7. Case when linear factors are not possible.

If $F(D, D')$ cannot be factorised into linear factors, the method illustrated by the following examples is to be followed :—

$$(D-D'^2)\, z=\cos(x-3y).$$

Now for complementary function, we have

$$(D-D'^2)\, z=0. \qquad \ldots(1)$$

Assume that $z=Ae^{hx+ky}$ is a solution.

Substituting in (1), $(Ahe^{kx+ky}-Ae^{hx+ky}.k^2)=0$

or $$h-k^2=0 \quad \text{or} \quad h=k^2.$$

$\therefore$ the complementary function is Ae^{k^2x+ky} where A and k are constants, more generally $\Sigma\, Ae^{k^2x+ky}$.

$$\text{P. I.}=\frac{\cos(x-3y)}{D-D'^2}=\frac{1}{D+9}\cos(x-3y)$$

$$=\frac{D-9}{D^2-81}\cos(x-3y)$$

$$=-\tfrac{1}{82}(D-9)\cos(x-3y)$$

$$=\tfrac{1}{82}\sin(x-3y)+\tfrac{9}{82}\cos(x-3y).$$

$\therefore$ the solution is

$$z=\Sigma\, Ae^{k(y+kx)}+\tfrac{1}{82}\sin(x-3y)+\tfrac{9}{82}\cos(x-3y).$$

2. $\dfrac{\partial^2 y}{\partial x^2}-\dfrac{\partial^2 y}{\partial z^2}=y+e^{z+x}$ or $(D^2-D'^2-1)\, y=e^{x+z}$,

where $D=\dfrac{\partial}{\partial x}$ and $D'=\dfrac{\partial}{\partial z}$.

For the complementary function,

$$(D^2-D'^2-1)y=0.$$

Let Ae^{hx+ky} be a solution. Substituting,

$$Ae^{hx+ky}\,(h^2-k^2-1)=0$$

or $$h^2-k^2-1=0.$$

Putting $h=\sec\alpha$, $k=\tan\alpha$, the complementary function is

$$Ae^{x\sec\alpha+z\tan\alpha},$$

where A and α are constants.

$$\text{P. I.}=\frac{1}{D^2-D'^2-1}\,e^{x+z}=-e^{x+z}.$$

$\therefore$ the general solution is

$$y=\Sigma\, Ae^{x \text{ sce } \alpha+z \tan \alpha}-e^{x+z}.$$

Exercise 1 (5)

1. $(D^3-3DD'+D+1)\,z=e^{2x+3y}$.

Ans. $z=\Sigma\, Ae^{hx+y\,[(h^3+h+1)/3h]}-\frac{1}{7}e^{2x+3y}$.

2. $(D^2-DD'-2D)\,z=\sin(3x+4y)$.

Ans. $z=\Sigma\, Ae^{hx+(h-2)y}+\frac{1}{15}\,[\sin(3x+4y)+2\cos(3x+4y)$.

1·8 Working rule for solving equations reducible to the linear form, namely

$$\Sigma A x^r\,\frac{\partial^r z}{\partial x^r}+\Sigma B_{p,\,q}x^p y^q\,\frac{\partial^{p+q}z}{\partial x^p\,\partial y^q}+\Sigma C_s y^s\,\frac{\partial^s z}{\partial y^s}=0.$$

Replace $x^r\dfrac{\partial^r z}{\partial x^r}$ by $D\,(D-1)\,(D-2)\ldots(D-r+1)\,z$,

$x^p y^q\dfrac{\partial^{p+q}z}{\partial x^p\,\partial y^q}$ by $D\,(D-1)\ldots(D-p+1)\,D'\,(D'-1)\ldots$

$\ldots(D'-q+1)\,z$

and $y^s\dfrac{\partial^s z}{\partial y^s}$ by $D'\,(D'-1)\,(D'-2)\ldots(D'-s+1)\,z$,

where $x=e^u$, and therefore $x\dfrac{\partial}{\partial x}=\dfrac{\partial}{\partial u}=D$ (say),

$y=e^v$, and therefore $y\dfrac{\partial}{\partial y}=\dfrac{\partial}{\partial v}=D'$ (say).

The equation will then be linear with u and v as the new independent variables.

Solve all examples 1 (4) and exercises 1 (4) by this working rule.

CHAPTER II

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND AND HIGHER ORDERS *(Continued)*

2·0. We shall usually take z as dependent and x, y as independent variables and throughout this chapter we shall denote

$$\frac{\partial z}{\partial x} \text{ by } p, \ \frac{\partial z}{\partial y} \text{ by } q, \ \frac{\partial^2 z}{\partial x^2} \text{ by } r, \ \frac{\partial^2 z}{\partial x \, \partial y} \text{ by } s \text{ and } \frac{\partial^2 z}{\partial y^2} \text{ by } t.$$

2·1. Monge's method of solving the equation

$$Rr+Ss+Tt=V \qquad \ldots(1)$$

where r, s, t, have their usual meanings and R, S, T and V are functions of x, y, z, p and q. **(Agra 1963, '56, '54, '52)**

We know
$$dp=\frac{\partial p}{\partial x}\,dx+\frac{\partial p}{\partial y}\,dy$$
$$=r\,dx+s\,dy$$

and
$$dq=\frac{\partial q}{\partial x}\,dx+\frac{\partial q}{\partial y}\,dy,$$
$$=s\,dx+t\,dy.$$

Putting the values of r and t in (1),

$$R\left(\frac{dp-s\,dy}{dx}\right)+S.s+T.\left(\frac{dq-s\,dx}{dy}\right)=V$$

or $\quad R\,dp\,dy+T\,dq\,dx+Ss\,dx\,dy-Rs\,dy^2-Ts\,dx^2=V\,dx\,dy$

or $\quad (R\,dp\,dy+T\,dq\,dx-V\,dx\,dy)$
$$=s\,(R\,dy^2-S\,dx\,dy+T\,dx^2). \qquad \ldots(2)$$

If some relation between x, y, z, p, q makes each of the bracketed expressions vanish, the relation will satisfy (2); therefore

$$R\,dy^2-S\,dx\,dy+T\,dx^2=0 \qquad \ldots(3)$$

$$R\,dp\,dy+T\,dq\,dx-V\,dx\,dy=0. \qquad \ldots(4)$$

Now it may be possible to get one or two relations between x, y, z, p, q called intermediate integrals, and then to find the general solution of (1).

If (3) resolves into two linear equations in dx and dy such as

$$dy-m_1\,dx=0 \text{ and } dy-m_2\,dx=0, \qquad \ldots(5)$$

from one of the equations (5) combined with (4) and if necessary with $dz=p\,dx+q\,dy$, we may obtain two integrals $u_1=a$ and $v_1=b$; then

$$u_1=f_1\,(v_1),$$

where f_1 is an arbitrary function, u_1 is an intemediate integral.

Proceeding similarly from the second equation, we may get another intermediate integral $u_2=f_2\,(v_2)$.

From these two integrals we may find the values of p and q and putting these values in $dz=p\,dx+q\,dy$ and integrating it we get the complete integral of the original equation.

If $m_1=m_2$, either of the intermediate integrals may be integrated to give the solution of the given equation.

Examples 2 (1)

1. *Solve* $r=a^2t$. **(Agra 1962, 1959)**

(This can be easily solved by the method discussed in the last chapter. Here we solve it by Monge's Method.)

Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$ in the given equation,

$$dp\,dy-a^2\,dx\,dq=s\,(dy^2-a^2\,dx^2).$$

So the subsidiary equations are

$$dy^2-a^2\,dx^2=0 \qquad \ldots(1)$$

and

$$dp\,dy-a^2\,dx\,dq=0. \qquad \ldots(2)$$

From (1), $$dy+a\,dx=0, \qquad \ldots(3)$$
$$dy-a\,dx=0. \qquad \ldots(4)$$

Taking (3) and combining with (2), we get
$$dp+a\,dq=0.$$
$$p+qa=A.$$
Also $$y+ax=B.$$
$\therefore$ $p+aq=\phi_1(y+ax)$ is an intermediate integral.

Similarly $p-aq=\phi_2(y-ax)$ is the second intermediate integral.

From these,
$$p=\tfrac{1}{2}[\phi_1(y+ax)+\phi_2(y-ax)]$$
and $$q=\frac{1}{2a}[\phi_1(y+ax)-\phi_2(y-ax)].$$

Substituting these values in $dz=p\,dx+q\,dy$, we have,
$$dz=\tfrac{1}{2}[\phi_1(y+ax)+\phi_2(y-ax)]\,dx$$
$$+\frac{1}{2a}[\phi_1(y+ax)-\phi_2(y-ax)]\,dy$$

or, $$dz=\frac{1}{2a}(dy+a\,dx)\,\phi_1(y+ax)-\frac{dy-a\,dx}{2a}\,\phi_2(y-ax),$$

or, $$z=f_1(y+ax)+f_2(y-ax).$$

2. *Solve*
$$(b+cq)^2 r-2(b+cq)(a+cp)s+(a+cp)^2 t=0,$$
(Agra 1962, 1959, 1956)

Putting $r=\dfrac{dp-s\,dy}{dx}$, $t=\dfrac{dq-s\,dx}{dy}$,
$$(b+cq)^2\frac{dp-s\,dy}{dx}-2(b+cq)(a+cp)s$$
$$+(a+cp)^2\frac{dq-s\,dx}{dy}=0.$$

$\therefore$ the subsidiary equations are,
$$(b+cq)^2\,dy^2+2(b+cq)(a+cp)\,dx\,dy+(a+cp)^2\,dx^2=0, \qquad \ldots(1)$$
$$(b+cq)^2\,dp\,dy+(a+cp)^2\,dq\,dx=0. \qquad \ldots(2)$$
From (1), $$(b+cq)\,dy+(a+cp)\,dx=0 \qquad \ldots(3)$$

Combining it with (2),

$$(b+cq)\,dp-(a+cp)\,dq=0$$

from which $$\frac{dp}{a+cp}=\frac{dq}{b+cq}$$

and therefore, $$(a+cp)=A\,(b+cq). \qquad \ldots(4)$$

Also from (3) and $dz=p\,dx+q\,dy$, we get

$$a\,dx+b\,dy+c\,dz=0$$

or $$ax+by+cz=B. \qquad \ldots(5)$$

$\therefore$ from (4) and (5),

$$a+cp=(b+cq)\,\phi\,(ax+by+cz).$$

$$\therefore \quad \frac{dx}{c}=\frac{dy}{-c\phi}=\frac{dz}{-a+b\phi}=\frac{a\,dx+b\,dy+c\,dz}{0} \qquad \ldots(6)$$

where ϕ stands for $\phi\,(ax+by+cz)$,

so that $$ax+by+cz=K_1$$

and $$\frac{dx}{c}=\frac{dy}{-c\phi\,(K_1)}.$$

Integrating, $$x\phi\,(K_1)=-y+K_2.$$

$\therefore$ $y+x\phi\,(ax+by+cz)=\psi\,(ax+by+cz)$. [as $K_2=\psi\,(K_1)$].

3. *Solve* $r+(a+b)\,s+abt=xy$. **(Agra 1955, 58)**

Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$,

$$\frac{dp-s\,dy}{dx}+(a+b)\,s+ab\,\frac{dq-s\,dx}{dy}=xy$$

or $$dp\,dy+ab\,dq\,dx-xy\,dx\,dy$$
$$=s\,[dy^2-(a+b)\,dx\,dy+ab\,dx^2].$$

$\therefore$ the subsidiary equations are

$$dy^2-(a+b)\,dx\,dy+ab\,dx^2=0 \qquad \ldots(1)$$

and $$dp\,dy+ab\,dq\,dx=xy\,dx\,dy=0. \qquad \ldots(2)$$

From (1), $$dy-a\,dx=0, \qquad \ldots(3)$$
$$dy-b\,dx=0, \qquad \ldots(4)$$

whence $$y-ax=c_1, \quad \text{and} \quad y-bx=c_2.$$

Combining these with (2), we get

$$a\,dp+ab\,dq-ax\,(c_1+ax)\,dx=0$$

and

$$b\,dp+abq-bx\,(c_2+bx)\,dx=0.$$

or

$$\therefore\quad p+bq-c_1\frac{x^2}{2}-\frac{ax^3}{3}=A,$$

$$p+aq-c_2\frac{x^2}{2}-\frac{bx^3}{3}=B$$

or

$$p+bq-(y-ax)\frac{x^2}{2}-\frac{ax^3}{3}=\phi_1(c_1)=\phi_1(y-ax),$$

$$p+ap-(y-bx)\frac{x^2}{2}-\frac{bx^3}{3}=\phi_2(c_2)=\phi_2(y-bx).$$

Solving,

$$p=\frac{1}{a-b}\left[\frac{yx^2}{2}(a-b)-(a^2-b^2)\frac{x^3}{6}+a\phi_1(y-ax)-b\phi_2(y-bx)\right].$$

$$q=\frac{1}{b-a}\left[-\frac{x^3}{6}(a-b)+\phi_1(y-ax)-\phi_2(y-bx)\right].$$

Putting these values in $dz=p\,dx+q\,dy$,

$$dz=\left[\frac{yx^2}{2}-(a+b)\frac{x^3}{6}+\frac{a\phi_1(y-ax)}{a-b}-\frac{b\phi_2(y-bx)}{a-b}\right]dx$$

$$+\left[\frac{x^3}{6}-\frac{\phi_1(y-ax)}{a-b}+\frac{\phi_2(y-bx)}{a-b}\right]dy$$

$$=-\frac{(a+b)\,x^3}{6}\,dx+\frac{3x^2y\,dx+x^3\,dy}{6}$$

$$-\frac{1}{a-b}\left[\phi_1(y-ax)(dy-a\,dx)\right]$$

$$+\frac{1}{a-b}+\left[\phi_2(y-bx)(dy-bdx)\right]$$

$$\therefore\quad z=-\frac{(a+b)\,x^4}{24}+\frac{yx^3}{6}+\psi_1(y-ax)+\psi_2(y-bx).$$

Note. This question could be solved by the method of 1st chapter also.

4. *Solve $q(1+q)\,r-(p+q+2pq)\,s+p(1+p)\,t=0.$*

(Agra 1957)

Putting $r=\dfrac{dp-s\,dy}{dx}$, $t=\dfrac{dq-s\,dx}{dy}$,

$$(q+q^2)\,\frac{dp-s\,dy}{dx}-(p+q+2pq)\,s+p\,(1+p)\,\frac{dq-s\,dx}{dy}=0$$

or $$[(q+q^2)\,dp\,dy+(p+p^2)\,dq\,dx]$$
$$=s\,[(q+q^2)\,dy^2+(p+q+2pq)\,dx\,dy+(p+p^2)\,dx^2].$$

$\therefore$ the subsidiary equations are

$$(q+q^2)\,dp\,dy+p\,(1+p)\,dq\,dx=0 \qquad \ldots(1)$$

and $$[(q+q^2)\,dy^2+(p+q+2pq)\,dx\,dy+(p+p^2)\,dx^2]=0 \qquad \ldots(2)$$

From (2), $$q\,dy+p\,dx=0 \qquad \ldots(3)$$

and $$(1+q)\,dy+(1+p)\,dx=0. \qquad \ldots(4)$$

From (3) and $dz=p\,dx+q\,dy$, we have

$$dz=0, \quad \text{or,} \quad z=C_1 \qquad \ldots(5)$$

and from (4) and $dz=p\,dx+q\,dy$, we have

$$dx+dy+dz=0,$$

or, $$x+y+z=C_2. \qquad \ldots(6)$$

Now combining (3) with (1),

$$(q+1)\,dp-(p+1)\,dq=0. \qquad \ldots(7)$$

and combining (4) with (1),

$$q\,dp-p\,dq=0 \qquad \ldots(8)$$

i.e. $$dp-dq=0 \qquad \text{[from (7) and (8)]}$$

or $$p-q=k_1=\phi_1\,(C_1)=\phi_1\,(z).$$

$$\therefore \quad \frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{\phi_1\,(z)}$$

or $$x=F_1\,(z)+k_2=F_1\,(z)+F_2\,(C_2)$$
$$=F_1\,(z)+F_2\,(x+y+z).$$

5. *Solve* $q^2r-2pqs+p^2t=0$ *and show that the integral represents a surface generated by straight lines which are parallel to a fixed plane.* **(Agra, 1961)**

Putting $r=\dfrac{dp-s\,dy}{dx}$, and $t=\dfrac{dq-s\,dx}{dy}$,

$$(q^2\,dp\,dy+p^2\,dq\,dx)=s\,(q^2\,dq^2+2pq\,dx\,dy+p^2\,dx^2).$$

$\therefore$ the subsidiary equations are

$$q^2\, dp\, dy + p^2\, dq\, dx = 0 \qquad \ldots(1)$$

$$q\, dy + p\, dx = 0. \qquad \ldots(2)$$

Also
$$dz = p\, dx + q\, dy = 0.$$

$$\therefore \quad z = c.$$

From (1) and (2),

$$q\, dp - p\, dq = 0$$

or
$$p/q = k = f(c)$$

or
$$p - q\, f(c) = 0.$$

$$\therefore \quad \frac{dx}{1} = \frac{dy}{-f(c)} = \frac{dz}{0},$$

$$y + x\, f(c) = K = F(c)$$

or,
$$y + x\, f(z) = F(z). \qquad \ldots(3)$$

The integral of the differential equation is the surface (3) which is the locus of the straight lines given by the intersections of planes $y + xf(c) = F(c)$, and $z = c$. These lines are all parallel to the plane $z = 0$ as they lie on the plane $z = c$ for varying values of c.

6. *Solve* $r - a^2 t + 2ab(p + qa) = 0$.

Putting $r = \dfrac{dp - s\, dy}{dx}$ and $t = \dfrac{dq - s\, dx}{dy}$, we get

$$dp\, dy - a^2\, dq\, dx + 2ab(p + aq)\, dx\, dy = s(dy^2 - a^2\, dx^2)$$

$\therefore$ the subsidiary equations are

$$dy^2 - a^2\, dx^2 = 0, \qquad \ldots(1)$$

$$dp\, dy - a^2\, dq\, dx + 2ab(p + qa)\, dx\, dy = 0. \qquad \ldots(2)$$

From (1),
$$y + ax = \alpha, \qquad \ldots(3)$$

$$y - ax = \beta. \qquad \ldots(4)$$

From (3) and (2),

$$dp + a\, dq + 2ab(p + qa)\, dx = 0$$

or
$$\frac{dp + a\, dq}{p + aq} = -2ab\, dx.$$

$$\therefore \quad \log(p + qa) = -2abx + \log c,$$

$$\therefore \quad \frac{p+aq}{c}=\frac{(p+aq)}{f(\alpha)}=e^{-2abx}$$

or $$p+qa=f(\alpha)\, e^{-2abx}. \qquad \ldots(5)$$

$$\therefore \quad \frac{dx}{1}=\frac{dy}{a}=\frac{dz}{f(\alpha)\, e^{-2abx}}.$$

Integrating, $$\frac{f(\alpha)\, e^{-2abx}}{-2ab}=z+k=z+\phi(\beta)$$

or, $$z=f_1(y+ax)\, e^{-2abx}+f_2(y-ax).$$

7. *Solve* $r-t\cos^2 x+p\tan x=0.$

Putting $r=\dfrac{dp-s\,dy}{dx}$, $t=\dfrac{dq-s\,dx}{dy}$, we get

$$dp\,dy-\cos^2 x\,dx\,dq+p\tan x\,dx\,dy$$
$$=s\,(dy^2-\cos^2 x\,dx^2).$$

$\therefore$ the subsidiary equations are

$$dy^2-\cos^2 x\,dx^2=0, \qquad \ldots(1)$$

$$dp\,dy-\cos^2 x\,dx\,dq+p\tan x\,dx\,dy=0. \qquad \ldots(2)$$

From (1), $$y=\sin x+\alpha, \qquad \ldots(3)$$

$$y=-\sin x+\beta. \qquad \ldots(4)$$

From (2) and (3),

$$\cos x\,dp-\cos^2 x\,dq+p\sin x\,dx=0$$

or $$\sec x\,dp-dq+p\tan x\sec x\,dx=0$$

or $$p\sec x-q=c_1=f(\alpha)$$
$$=f(y-\sin x).$$

$$\therefore \quad \frac{dx}{\sec x}=\frac{dy}{-1}=\frac{dz}{f(y-\sin x)}.$$

and hence, $$f(y-\sin x)\,\frac{(dy-\cos x\,dx)}{2}=-dz.$$

$$\therefore \quad F(y-\sin x)+2z=c_2=G(\beta).$$

$$\therefore \quad F(y-\sin x)+2z=G(y+\sin x). \qquad \text{[from (4)]}$$

8. *Solve* $(x-y)(xr-sx-ys+yt)=(x+y)(p-q).$

(Agra 1963, 1954)

Putting $r=\dfrac{dp-s\,dy}{dx}$ and $t=\dfrac{dq-s\,dx}{dy}$, we get

$$(x-y)\left[x\,\frac{dp-s\,dy}{dx}-sx-ys+y\,\frac{dq-s\,dx}{dy}\right]=(x+y)\,(p-q)$$

or $\quad x\,(x-y)\,dp\,dy+y\,(x-y)\,dq\,dx-(x+y)\,(p-q)\,dx\,dy$
$\quad =s\,[x\,(x-y)\,dy^2+(x+y)\,(x-y)\,dx\,dy+y\,(x-y)\,dx^2]$.

$\therefore$ the subsidiary equations are

$$x\,dy^2+(x+y)\,dx\,dy+y\,dx^2=0 \qquad \ldots(1)$$

or $\quad x\,(x-y)\,dp\,dy+y\,(x-y)\,dq\,dx-(x+y)\,(p-q)\,dx\,dy=0.$ $\qquad \ldots(2)$

From (1), $\quad x\,dy+y\,dx=0 \qquad \ldots(3)$

and $\quad dy+dx=0 \qquad \ldots(4)$

or $\quad xy=\alpha$ and $y+x=\beta$. $\qquad \ldots(4')$

From (3) and (2),

$$-y\,(x-y)\,dp+y\,(x-y)\,dq-(p-q)\,[-y\,dx+y\,dy]=0$$

or $\quad (x-y)\,(dp-dq)=(p-q)\,(dx-dy)$

or $\quad \dfrac{dp-dq}{p-q}=\dfrac{dx-dy}{x-y}$

or $\quad \log\,(p-q)=\log\,[(x-y).k]$.

hence, using (4'), $\quad (p-q)=(x-y)\,f\,(xy). \qquad \ldots(5)$

From (4) and (2),

$$-x\,(x-y)\,dp+y\,(x-y)\,dq-(p-q)\,(x\,dy-y\,dx)$$

or $\quad (x-y)\,(x\,dp-y\,dq)=-[\,px\,dy-qx\,dy-py\,dx+qy\,dx]$
$\quad =(px-qy)\,dx-(px-qy)\,dy-px\,dx-qy\,dy$
$\qquad +py\,dx+qx\,dy$
$\quad =-(px-qy)\,(dy-dx)-(p\,dx-q\,dy)\,x+(p\,dx-q\,dy)\,y$

or $\quad (x-y)\,(x\,dp-y\,dq+p\,dx-q\,dq)=(px-qy)\,(dx-dy)$

or $\quad \dfrac{x\,dp+p\,dx-y\,dy-q\,dy}{px-qy}=\dfrac{dx-dy}{x-y}.$

$\therefore \quad \log\,(px-qy)=\log\,C\,(x-y)$.

Hence, using (4'), $px-qy=(x-y)\,\phi\,(x+y). \qquad \ldots(6)$

Solving (5) and (6),

$$p=\phi\,(x+y)-y\,f\,(xy),$$
$$q=\phi\,(x+y)-x\,f\,(xy).$$

Now $dz=p\,dx+q\,dy$

$$=[\phi(x+y)-y\,f(xy)]\,dx+[\phi(x+y)-x\,f(xy)]\,dy$$
$$=\phi(x+y)(dx+dy)-f(xy)[y\,dx+x\,dy].$$
$$\therefore \quad z=\psi_1(x+y)+\psi_2(xy).$$

Exercises 2 (1)

1. $x^2r+2xys+y^2t=0.$ **Ans.** $z=f(y/x)+xF(y/x)$
2. $x^2r-y^2t=0.$ **Ans.** $z=x\,f(y/x)+F(xy)$
3. $r-2as+a^2t=0.$ **Ans.** $z=x\,f(y+ax)+F(y+ax)$
4. $2x^2r-5xys+2y^2t+2(px+qy)=0.$
   Ans. $z=f(x^2y)+F(xy^2)$
5. $y^2r-2ys+t=p+6y.$ **Ans.** $z=y^3-y\phi(2x+y^2)+f(2x+y^2)$
6. $pt-qs=q^3.$ **Ans.** $y=zx+f(z)+F(x)$
7. $(1+q)^2r-2(1+p+q+pq)s+(1+p)^2t=0.$
   Ans. $y=f(x+y+z)+x\,F(x+y+z)$
8. $xy(t-r)+(x^2-y^2)(s-2)=py-qx.$
   Ans. $z=f(x^2+y^2)+F(y/x)+xy$

2·11. When, in the Monge's form of the equation

$$Rr+Ss+Tt=V,$$

R, S, T are constants the equation can be better solved by methods of Chapter 1.

Examples 2 (2)

1. *Solve* $r+s-6t=y\cos x.$ **(Agra 1961, 1963)**

We have

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\,\partial y}-6\frac{\partial^2 z}{\partial y^2}=y\cos x$$

or
$$(D^2+DD'-6D'^2)\,z=y\cos x$$

or
$$(D+3D')(D-2D')\,z=y\cos x.$$

The complementary function is obviously

$$\phi_1(y-3x)+\phi_2(y+2x). \qquad \ldots(1)$$

The particular integral is

$$\frac{y\cos x}{D^2+DD'-6D'^2}$$

$$=\frac{y\cos x}{-1+DD'-6D'^2}$$

$$=\frac{\{(-1-6D'^2)-DD'\}\,y\cos x}{(-1-6D'^2)^2-D^2D'^2}$$

$$=\frac{-y\cos x-D\,(\cos x)}{(1+6D'^2)^2-D^2D'^2}$$

$$=\frac{-y\cos x+\sin x}{(1+6D'^2)^2-D^2D'^2}$$

$$=\frac{-y\cos x+\sin x}{(1+6D'^2)^2+D'^2}$$

$$=(1+13D'^2+36D'^4)^{-1}\,(-y\cos x+\sin x)$$

$$=-y\cos x+\sin x. \qquad \ldots(2)$$

Hence adding (1) and (2), the complete solution is

$$z=\phi_1\,(y-3x)+\phi_2\,(y+2x)-y\cos x+\sin x.$$

2. *Solve* $4\,(r-s)+t=16\log\,(x+2y)$. **(Agra 1960)**

We have $4\left(\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\,\partial y}\right)+\frac{\partial^2 z}{\partial y^2}=16\log\,(x+2y)$

or $$(4D^2-4DD'+D'^2)\,z=16\log\,(x+2y)$$

or $$(2D-D')^2\,z=16\log\,(x+2y).$$

The complementary function is obviously

$$z=\phi_1\,(2y+x)+x\,\phi_2\,(2y+x). \qquad \ldots(1)$$

The particular integral is

$$z=\frac{16\log\,(x+2y)}{(2D-D')^2}$$

$$=16.\frac{x^2}{2^2\,(2\,!)}\log\,(x+2y)$$

$$=2x^2\log\,(x+2y) \quad [(F)\text{ of II Art. } 1\cdot33] \qquad \ldots(2)$$

Thus by (1) and (2), the complete solution is,

$$z=\phi_1\,(2y+x)+x\,\phi_2\,(2y+x)+2x^2\log\,(x+2y).$$

Exercises 2 (2)

Solve :

1. $r+(a+b)\,s+abt=xy.$ **(Agra 1958)**

Ans. $z=\phi_1(y-ax)+\phi_2(y-bx)+\dfrac{x^3y}{6}-(a+b)\dfrac{x^4}{24}.$

2. $r-6s+9t=12x^2+36xy.$

Ans. $z=\phi_1(y+3x)+x\phi_2(y+3x)+10x^4+6x^3y.$

3. $r-s-6t=xy.$ Ans. $z=\phi_1(y+3x)+\phi_2(y-2x)+\dfrac{x^3y}{6}+\dfrac{x^4}{24}.$

4. $r-2s+t=12xy.$ Ans. $z=\phi_1(y+x)+x\phi_2(y+x)+2x^3y+x^4.$

5. $\log s=x+y$ (or $s=e^{x+y}$). Ans. $z=\phi_1(x)+\phi_2(y)+e^{x+y}.$

6. $r-a^2t=x.$ Ans. $z=\phi_1(y+ax)+\phi_2(y-ax)+(x^3/6).$

7. $r-t=x-y.$ Ans. $\phi_1(y+x)+\phi_2(y-x)+\dfrac{x^3}{6}-\tfrac{1}{2}x^2y.$

8. $r-2s+t=\sin(2x+3y).$

Ans. $z=\phi_1(y+x)+x\phi_2(y+x)-\sin(2x+3y).$

9. $r+3s+2t=x+y.$ **(Agra 1952)**

Ans. $z=\phi_1(y-x)+\phi_2(y-2x)+\dfrac{(x+y)^3}{36}.$

10. $2r-5s+2t=24(x-y).$

Ans. $z=\phi_1(y+2x)+\phi_2(2y+x)+\tfrac{4}{9}(y-x)^3.$

11. $r+t=30(2x+y).$

Ans. $z=\phi_1(y+ix)+\phi_2(y-ix)+(2x+y)^3.$

12. $r-5s+4t=\sin(4x+y).$

Ans. $z=\phi_1(y+x)+\phi_2(y+4x)+\dfrac{x\cos(4x+y)}{3}.$

13. $r-6s+9t=6x+2y.$

Ans. $z=\phi_1(y+3x)+x\phi_2(y+3x)+x^2(3x+y).$

14. $r-2as+a^2t=f(y+ax).$

Ans. $z=\phi_1(y+ax)+x\phi_2(y+ax)+\dfrac{x^2}{2}f(y+ax).$

15. $r-2s+t=x+\phi(x+y)$.

Ans. $z=\phi_1(y+x)+x\,\phi_2(y+x)+\dfrac{x^3}{6}+\dfrac{x^2\phi(y+x)}{2}$.

16. $r+2s+t=2\cos y-x\sin y$.

Ans. $z=\phi_1(y-x)+x\,\phi_2(y-x)+x\sin y$.

17. $r-s-2t=(y-1)\,e^x$. Ans. $z=\phi_1(y+2x)+\phi_2(y-x)+ye^x$.

2·2. Monge's method of integrating

$$Rr+Ss+Tt+U(rt-s^2)=V, \qquad \ldots(\text{F})$$

where R, S, T, U, V are functions of x, y, z, p, q.

As before, put, $\quad r=(dp-s\,dy)/dx$

and $\quad t=(dq-s\,dx)/dy$.

The equation reduces to

$$R\,dp\,dy+T\,dq\,dx+U\,dp\,dq-V\,dx\,dy$$
$$-s\,(R\,dy^2-S\,dx\,dy+T\,dx^2+U\,dp\,dx+U\,dq\,dy)=0$$

or $$N-Ms=0.$$

So far we used to factorise M, but on account of the presence of $U\,dp\,dx+V\,dq\,dy$, the factors are not possible; so let us try to factorise $M+\lambda N$, when λ is some multiplier to be determined later.

Now $\lambda N+M$

$$=\lambda\,(R\,dp\,dy+T\,dq\,dx+U\,dp\,dq-V\,dx\,dy)$$
$$+(R\,dy^2-S\,dx\,dy+T\,dx^2+U\,dp\,dx+U\,dq\,dy)$$
$$=R\,dy^2+T\,dx^2-(S+\lambda V)\,dx\,dy+U\,dp\,dx$$
$$+U\,dq\,dy+\lambda R\,dp\,dy+\lambda T\,dq\,dx+\lambda U\,dp\,dq.$$

Let the factors of the above be

$$\alpha\,dy+\beta\,dx+\gamma\,dp \text{ and } \alpha'\,dy+\beta'\,dx+\gamma'\,dq.$$

Equating coefficients of dy^2, dx^2, $dp\,dq$ in the product,

$$\alpha\alpha'=R,\ \beta\beta'=T,\ \gamma\gamma'=\lambda U.$$

Now if we take

$$\alpha=R,\ \alpha'=1,\ \beta=kT,\ \beta'=(1/k),\ \gamma=mU,\ \gamma'=\lambda/m,$$

equating the coefficients of the other five terms,

$$kT+R/k=-(S+\lambda V), \qquad \ldots(1)$$

$$\lambda R/m=U, \qquad \ldots(2)$$
$$kT\lambda/m=\lambda T, \qquad \ldots(3)$$
$$mU=\lambda R, \qquad \ldots(4)$$
$$mU/k=U. \qquad \ldots(5)$$

From (5), $m=k$ and this satisfies (3).

From (2) and (3), $m=\lambda R/U=k$.

$\therefore$ from (1), $\lambda^2 (\mathbf{RT+UV})+\lambda \mathbf{US}+\mathbf{U}^2=0. \qquad \ldots(6)$

The first step in practical working is to form the equation (6) in λ and to determine the two roots λ_1 and λ_2 of this equation.

So if λ_1 is a root of (6), the factors of $M+\lambda N$ are

$$\left(R\,dy+\lambda_1 \frac{RT}{U}\,dx+\lambda_1 R\,dp\right)\left(dy+\frac{U}{\lambda_1 R}\,dx+\frac{U}{R}\,dq\right)$$

or $\dfrac{R}{U}(U\,dy+T\lambda_1\,dx+\lambda_1 U\,dp)\,\dfrac{1}{\lambda_1 R}(\lambda_1 R\,dy+U\,dx+\lambda_1 U\,dq)$.

Similarly if λ_2 is a root of (6), the factors corresponding to it are

$$\frac{R}{U}(U\,dy+T\lambda_2\,dx+\lambda_2 U\,dp)\times\frac{1}{\lambda_2 R}(\lambda_2 R\,dy+U\,dx+\lambda_2 U\,dq).$$

Now we may obtain two integrals $u_1=a_1$, $v_1=b_1$ of the equations

$$\left.\begin{aligned}&\mathbf{U\,dy+\lambda_1 T\,dx+\lambda_1 U\,dp=0}\\ \text{and}\quad &\mathbf{U\,dx+\lambda_2 R\,dy+\lambda_2 U\,dq=0}\end{aligned}\right\} \qquad \ldots(7)$$

or we may obtain two integrals $u_2=a_2$, $v_2=b_2$ of the equations

$$\left.\begin{aligned}&\mathbf{U\,dy+\lambda_2 T\,dx+\lambda_2 U\,dp=0,}\\ &\mathbf{U\,dx+\lambda_1 R\,dy+\lambda_1 U\,dq=0.}\end{aligned}\right\} \qquad \ldots(8)$$

Sets of equations (7) and (8), when written down, constitute the second important step in the solution of the given equation (F).

We thus get two intermediate integrals $u_1=f_1(v_1)$ and $u_2=f_2(v_2)$ and substituting in $dz=p\,dx+q\,dy$, the values of p and q obtained from the two intermediate integrals, we get the solution after integrating.

2·3. In case the two roots of the equation (6) in § 2·2 are equal, we shall get only intermediate integral $u_1=f(v_1)$ which together with one of the integrals $u_1=a_1$ and $v_1=b_1$ will give values of p and q suitable to solve $dz=p\,dx+q\,dy$.

2·4. If it is not possible to obtain the values of p and q from the two intermediate integrals $u_1=f_1(v_1)$ and $u_2=f_2(v_2)$, suitable for intertion in $dz=p\,dx+q\,dy$, we may take one of the intermediate integrals say $u_1=f_1(v_1)$ and one of the integrals from $u_2=a_2$ and $v_2=b_2$.

The values of p and q obtained from these and substituted in $dz=p\,dx+q\,dy$ will give the solution of the given equation.

Examples 2 (2)

1. *Solve $ar+bs+ct+e(rt-s^2)=h$ where a, b, c, e and h are constants.* **(Agra 1952)**

Here $R=a,\ S=b,\ T=c,\ U=e,\ V=h.$

The equation in λ is

$$\lambda^2(ac+eh)+\lambda be+e^2=0. \qquad \ldots(1)$$

Putting $$\lambda=-e/m, \qquad \ldots(2)$$

(1) becomes $$\frac{e^2}{m^2}(ac+eh)-\frac{e^2b}{m}+e^2=0$$

or $$m^2-bm+(ac+eh)=0 \qquad \ldots(3)$$

If m_1, m_2 are the roots of (3), the first system of intermediate integrals is given by

$$U\,dy+\lambda_1T\,dx+\lambda_1U\,dp=0,$$
$$U\,dx+\lambda_2R\,dy+\lambda_2U\,dq=0,$$

i.e. by $$e\,dy+\left(-\frac{e}{m_1}\right)c\,dx+\left(-\frac{e}{m_1}\right)e\,dp=0,$$
$$e\,dx+\left(-\frac{e}{m_2}\right)a\,dy+\left(-\frac{e}{m_2}\right)e\,dq=0$$

or by
$$c\,dx+e\,dp-m_1\,dy=0,$$
$$a\,dy+e\,dq-m_2\,dx=0\,;$$
so one of the intermediate integrals is
$$cx+ep-m_1y=f\,(ay+eq-m_2x). \qquad \ldots(4)$$
Similarly the second intermediate integral is
$$(cx+ep-m_2y)=F\,(ay+eq-m_1x). \qquad \ldots(5)$$
It is not possible to get the values of p and q from (4) (5) ; so we combine (4) with $cx+ep-m_2y=A$.

Thus we have
$$(m_2-m_1)\,y+A=f\,(ay+eq-m_2x)$$
or
$$ay+eq=m_2x+\phi\,[(m_2-m_1)\,y+A]$$
where ϕ is inverse function of f.

This gives q, and $cx+ep-m_2y=A$ gives p.

Substituting these values in $dz=p\,dx+q\,dy$,
$$e\,dz=(A-cx+m_2y)\,dx+[-ay+m_2x+\phi\,\{(m_2-m_1)\,y+A\}]\,dy.$$
Integrating,
$$ez+\frac{cx^2}{2}+\frac{ay^2}{2}=m_2xy+Ax+\psi\,\{(m_2-m_1)\,y+A\}+B$$
where
$$\psi\,(t)=\frac{\int\phi\,(t)\,dt}{m_2-m_1}.$$

2. *Solve*
$$z\,(1+q^2)\,r-2pqzs+z\,(1+p^2)\,t-z^2\,(s^2-rt)+1+p^2+q^2=0.$$
(Agra 1953)

Here $R=z\,(1+q^2)$, $S=-2pqz$, $T=(1+p^2)\,z$,
$$U=z^2,\; V=-(1+p^2+q^2).$$
The equation in λ is
$$(RT+UV)\,\lambda^2+\lambda US+U^2=0$$
or
$$z^2\lambda^2p^2q^2-2\lambda z^3pq+z^4=0$$
or
$$p^2q^2\lambda^2-2z\lambda pq+z^2=0$$
or
$$\lambda=z/pq. \qquad \text{(roots are equal)}.$$
$\therefore$ the system of intermediate integrals is given by
$$U\,dy+\lambda T\,dx+\lambda U\,dp=0,$$
$$U\,dx+\lambda R\,dy+\lambda U\,dq=0,$$

i. e., by $$pq\ dy+(1+p^2)\ dx+z\ dp=0, \qquad \ldots(1)$$
$$pq\ dx+(1+q^2)\ dy+z\ dq=0. \qquad \ldots(2)$$

Also $$dz=p\ dx+q\ dy. \qquad \ldots(3)$$

We write (1) as
$$dx+p\ (p\ dx+q\ dy)+z\ dp=0,$$

With the help of (3), it reduces to
$$dx+p\ dz+z\ dp=0$$
or
$$x+pz=\alpha.$$

Similarly from (2) and (3),
$$y+zq=\beta.$$

Putting the values of p and q in $dz=p\ dx+q\ dy$,
$$dz=\frac{\alpha-x}{z}\ dx+\frac{\beta-y}{z}\ dy$$
or
$$-z\ dz=(\alpha-x)\ (-dx)+(\beta-y)\ (-dy)$$
or
$$-\frac{z^2}{2}=\frac{(\alpha-x)^3}{2}+\frac{(\beta-y)^2}{2}+k$$
or
$$z^2+(x-\alpha)^2+(y-\beta)^2=\gamma^2$$
where a, β, γ are constants.

3. *Solve* $(1+q^2)\,r-2pqs+(1+p^2)\,t$
$$+(1+p^2+q^2)^{-1/2}\ (rt-s^2)=-(1+p^2+q^2)^{3/2}.$$

Here $R=(1+q^2)$, $S=-2pq$, $T=(1+p^2)$,
$U=(1+p^2+q^2)^{-1/2}$, $V=-(1+p^2+q^2)^{3/2}$.

The equation in λ is
$$(RT+UV)\ \lambda^2+\lambda US+U^2=0$$
or
$$[(1+p^2)\ (1+q^2)-(1+p^2+q^2)]\ \lambda^2 + \lambda\ \frac{(-2pq)}{\sqrt{(1+p+q^2)}}+\frac{1}{1+p^2+q^2}=0$$
or
$$\lambda^2p^2q^2\ (1+p^2+q^2)-2pq\sqrt{(1+p^2+q^2)}\ \lambda+1=0$$
or
$$\lambda=\frac{1}{pq\sqrt{(1+p^2+q^2)}}\ \text{(roots being equal).}$$

We get only one system which will give only one intermediate integral.

The system is

$$U\,dy+\lambda T\,dx+\lambda U\,dp=0,$$
$$U\,dx+\lambda R\,dy+\lambda U\,dq=0,$$

$$\frac{1}{\sqrt{(1+p^2+q^2)}}\,dy+\frac{(1+p^2)}{pq\sqrt{(1+p^2+q^2)}}\,dx+\frac{dp}{pq\,(1+p^2+q^2)}=0,$$
$$\frac{1}{\sqrt{(1+p^2+q^2)}}\,dx+\frac{(1+q^2)}{pq\sqrt{(1+p^2+q^2)}}\,dy+\frac{dq}{pq\,(1+p^2+q^2)}=0$$

or
$$pq\,dy+(1+p^2)\,dx+\frac{dp}{\sqrt{(1+p^2+q^2)}}=0,$$
$$pq\,dx+(1+q^2)\,dy+\frac{dq}{\sqrt{(1+p^2+q^2)}}=0.$$

Eliminating dy,

$$[(1+p^2)\,(1+q^2)-p^2q^2]\,dx$$
$$+[(1+q^2)\,dp-pq\,dq]/\sqrt{(1+p^2+q^2)}$$

or
$$dx+\frac{(1+q^2)\,dp-pq\,dq}{(1+p^2+q^2)^{3/2}}=0$$

or
$$dx+\frac{(1+p^2+q^2)\,dp}{(1+p^2+q^2)^{3/2}}-\frac{(p^2\,dp+pq\,dq)}{(1+p^2+q^2)^{3/2}}=0$$

or
$$dx+(1+p^2+q^2)^{-1/2}\,dp-\frac{\frac{1}{2}p\,(2p\,dp+2q\,dq)}{(1+p^2+q^2)^{3/2}}=0$$

or
$$x+p\,(1+p^2+q^2)^{-1/2}=\alpha. \qquad \ldots(1)$$

Similarly eliminating dx,

$$y+q\,(1+p^2+q^2)^{-1/2}=\beta. \qquad \ldots(2)$$

From (1) and (2),

$$\frac{(x-\alpha)}{(y-\beta)}=\frac{p}{q}. \qquad \ldots(3)$$

Substituting in (1), the value of p as found from (3),

$$q=\frac{y-\beta}{\sqrt{[1-\{(x-\alpha)^2+(y-\beta)^2\}]}}.$$

Similarly from (3) and (2),

$$p=\frac{x-\alpha}{\sqrt{[1-\{(x-\alpha)^2+(y-\beta)^2\}]}}.$$

Now, $$dz = p\,dx + q\,dy$$

or $$dz = \frac{(x-\alpha)\,dx + (y-\beta)\,dy}{\sqrt{[1-\{(x-\alpha)^2+(y-\beta)^2\}]}}.$$

Integrating, $$(z-\gamma) = -[1-\{(x-\alpha)^2+(y-\beta)^2\}]^{1/2}$$

or $$(z-\gamma)^2 = 1-[(x-\alpha)^2+(y-\beta)^2]$$

or $$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=1.$$

4. *Solve* $$s^2-rt=a^2$$

or $$rt-s^2=-a^2.$$

Here $R=0,\ S=0,\ T=0,\ U=1,\ V=-a^2.$

$\therefore$ the equation in λ is

$$\lambda^2\,(-\alpha^2)+\lambda.0+1=0$$

or $$\lambda=\pm 1/a.$$

The two intermediate integrals are given by

$$\left.\begin{aligned} -dy-\frac{1}{a}\,dp=0, \\ -dx+\frac{1}{a}\,dq=0. \end{aligned}\right\} \qquad \ldots(a)$$

$$\left.\begin{aligned} -dy+\frac{1}{a}\,dp=0, \\ -dx-\frac{1}{a}\,dq=0. \end{aligned}\right\} \qquad \ldots(b)$$

From (a), $$\left.\begin{aligned} p+ay=F(\alpha) \\ q-ax=\alpha \end{aligned}\right\} \qquad \ldots(c)$$

and from (b), $$\left.\begin{aligned} p-ay=F(\beta) \\ q+ax=\beta \end{aligned}\right\} \qquad \ldots(d)$$

i.e., the two intermediate integrals are

$$p+ay=f\,(q-ax) \qquad \ldots(1)$$

and $$p-ay=F\,(q+ax). \qquad \ldots(2)$$

Now since it is not possible to find the values of p and q from (1) and (2), we proceed as follows. Suppose α, β are not constants, but parameters.

Solving (c) and (d),

$$x=\frac{\beta-\alpha}{2a},\ q=\frac{\alpha+\beta}{2}, \qquad \ldots(3)$$

$$p=\tfrac{1}{2}\,[F(\alpha)+f(\beta)], \qquad \ldots(4)$$

$$y=\frac{1}{2a}\,[F(\alpha)-f(\beta)]. \qquad \ldots(5)$$

Substituting these values in $dz=p\,dx+q\,dy$,

$$dz=\frac{1}{4a}\,[F(\alpha)+f(\beta)]\,(d\beta-d\alpha)+\frac{\alpha+\beta}{4a}\,[F'(\alpha)\,d\alpha-f'(\beta)\,d\beta]$$

$$=\frac{1}{4a}\,[\{F(\alpha)\,d\beta+\beta F'(\alpha)\,d\alpha\}-\{f(\beta)\,d\alpha+\alpha f'(\beta)\,d\beta\}]$$

$$+\frac{1}{4a}\,[\{F(\alpha)\,d\alpha+\alpha F'(\alpha)\,d\alpha\}-\{f(\beta)\,d\beta+\beta f'(\beta)\,d\beta\}]$$

$$+\frac{1}{4a}\,[2f(\beta)\,d\beta-2F(\alpha)\,d\alpha].$$

$$\therefore\quad z=\frac{1}{4a}\,[\beta F(\alpha)-\alpha f(\beta)-\beta f(\beta)+\alpha F(\alpha)]$$

$$+\frac{2}{4a}\int f(\beta)\,d\beta-\frac{2}{4a}\int F(\alpha)\,d\alpha$$

$$=\frac{1}{4a}\,[F(\alpha)\,(\alpha+\beta)-f(\beta)\,(\alpha+\beta)]+\frac{2}{4a}\,G(\beta)-\frac{2}{4a}\,\phi(\alpha)$$

$$=\frac{\alpha+\beta}{2}\left[\frac{F(\alpha)-f(\beta)}{2a}\right]+\frac{1}{2a}\,G(\beta)-\frac{1}{2a}\,\phi(\alpha).$$

or $\quad z-qy=\psi_1\,(q+ax)+\phi_2\,(q-ax)\quad$ [from (3) and (5)]

where $$\psi_1\,(t)=\int\frac{f(t)}{2a}\,dt \qquad \ldots(6)$$

and $$\psi_2\,(t)=-\int\frac{F(t)}{2a}\,dt. \qquad \ldots(7)$$

Hence the primitive is

$$z-qy=\psi_1\,(q+ax)+\psi_2\,(q-ax)$$

$$-y=\psi_1'\,(q+ax)+\psi_2'\,(q-ax)\quad \text{[from (5), (6), and (7)]}.$$

5. *Solve*

$$qr+(p+x)\,s+yt+y\,(rt-s^2)=-$$

Here $R=q$, $S=(p+x)$, $T=y$, $U=y$, $V=-q$.

The equation in λ is

$$\lambda^2[qy-qy]+\lambda.(y)(p+x)+y^2=0$$

or, $$\lambda=\infty, \quad \text{or} \quad \lambda=-y/(p+x).$$

$\therefore$ the intermediate integrals are given by

$$\left.\begin{aligned} y\,dy-\frac{y^2}{p+x}\,dx-\frac{y^2}{p+x}\,dp=0 \\ \frac{y}{\infty}\,dx+q\,dy+y\,dq=0 \end{aligned}\right\}. \qquad \text{...(a)}$$

$$\left.\begin{aligned} y\,dx-\frac{qy}{p+x}\,dy-\frac{y^2\,dq}{p+x}=0 \\ \frac{y}{\infty}\,dy+y\,dx+y\,dp=0 \end{aligned}\right\}. \qquad \text{...(b)}$$

From (a), $$[(p+x)/y]=\alpha, \qquad \text{...(1)}$$

$$qy=F(\alpha) \qquad \text{...(2)}$$

or one of the integrals is

$$qy=F[(p+x)/y].$$

From second equation of (b),

$$p+x=\beta;\ \frac{p+x}{y}=\frac{\beta}{y}=\alpha \quad \text{[from (1)]} \qquad \text{...(2')}$$

or $$p=\beta-x \qquad \text{...(3)}$$

and from (2) and (1),

$$q=\frac{1}{y}F\left(\frac{p+x}{y}\right)=\frac{1}{y}F\left(\frac{\beta}{y}\right)=\frac{1}{y}F(\alpha) \quad \text{[from (2')]}$$

$$=\frac{\alpha}{\beta}.F(\alpha)\left[\because \text{ from (1) and (3), } \frac{1}{y}=\frac{\alpha}{\beta}\right] \qquad \text{...(4)}$$

Now, $$dz=p\,dx+q\,dy$$

$$=(\beta-x)\,dx+\frac{\alpha}{\beta}F(\alpha)\,dy \quad \text{[from (3) and (4)].}$$

$$\therefore \quad z=\beta x-\frac{x^2}{2}+\frac{\alpha}{\beta}F(\alpha)\,y+k$$

$$=\beta x-\frac{x^2}{2}+\frac{1}{y}F\left(\frac{\beta}{y}\right)y+\phi(\beta)$$

or,
$$z=\beta x-\frac{x^2}{2}+F\left(\frac{\beta}{y}\right)+\phi(\beta).$$

Exercises 2 (2)

Solve :—

1. $xqr+ypt+xy(s^2-rt)=pq$. **Ans.** $\alpha+z=F(\alpha)x^2+f(\alpha)y^2$.
2. $2s+(rt-s^2)=1$. **Ans.** $z=xy-k_1x+k_2y+c$.
3. $r+3s+t+(rt-s^2)=1$.
   Ans. $z=-\frac{1}{2}(x-y)^2-\phi(\alpha)+\psi(\beta)+\beta y$.
4. $3r+4s+t+(rt-s^2)=1$.
   Ans. $z=2xy-\frac{1}{2}(x^2+3y^2)+nx+\psi(y+mx)$

Finding the equation of a surface under given conditions.

Examples 2 (3)

1. *Find a surface passing through the parabolas*
$$z=0,\ y^2=4ax \text{ and } z=1,\ y^2=-4ax$$
and satisfying the equation
$$xr+2p=0.$$

Now
$$xr+2p=0$$
or
$$x\frac{\partial p}{\partial x}+2p=0 \quad \text{or,} \quad x^2\frac{\partial p}{\partial x}+2xp=0,$$
which gives
$$x^2p=f(y)$$
or
$$p=\frac{1}{x^2}f(y)$$
giving $z=-\frac{1}{x}f(y)+F(y)$, by integration.

Now this passes through $z=0$, $y^2=4ax$.
$$\therefore\quad 0=-\frac{4a}{y^2}f(y)+F(y).$$

Also it passes through $z=1$, $y^2=-4ax$.
$$\therefore\quad 1=-\frac{4a}{y^2}f(y)+F(y),$$
from which
$$F(y)=\tfrac{1}{2},\ f(y)=y^2/8a.$$

Hence the required equation is

$$z=\tfrac{1}{2}-\frac{y^2}{8ax}$$

or

$$8azx=4ax-y^2.$$

2. *Find a surface satisfying $r+s=0$ and touching the elliptic paraboloid $z=4x^2+y^2$ along its section by the plane $y=2x+1$.*

Now

$$r+s=0$$

i.e.

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\,\partial y}=0$$

or

$$D\,(D+D')\,y=0.$$

$\therefore$ the solution is

$$z=f\,(y-x)+F\,(y). \qquad \ldots(\alpha)$$

Now

$$p=\partial z/\partial x=-f'\,(y-x)$$

and

$$q=\partial z/\partial y=f'\,(y-x)+F'\,(y).$$

Also from

$$z=4x^2+y^2, \qquad \ldots(\text{a})$$

$$p=\partial z/\partial x=8x,$$

$$q=2y.$$

The values of p and q under the given condition for the two surfaces for any point on $y=2x+1$ should be equal.

Equating the two values of p and q for points on $y=2x+1$,

$$-f'\,(y-x)=8x, \qquad \ldots(1)$$

$$f'\,(y-x)+F'\,(y)=2y \qquad \ldots(2)$$

$$y=(2x+1) \qquad \ldots(3)$$

or

$$-f'\,(y-x)=8\,(y-x-1) \qquad [\text{from (1) and (3)}].$$

$$\therefore \quad -f\,(y-x)=8\,\frac{(y-x)^2}{2}-8\,(y-x)+K_1.$$

or

$$f\,(y-x)=-4x^2-4y^2+8xy+8y-8x-K_1. \qquad \ldots(4)$$

Also

$$-8x+F'\,(y)=2y \qquad [\text{from (1) and (2)}]$$

or

$$F'\,(y)=2y+8\left(\frac{y-1}{2}\right) \qquad [\text{from (3)}]$$

$$=6y-4.$$

$$\therefore \quad F(y)=3y^2-4y+K_2. \qquad \ldots(5)$$

$$z=-4x^2-4y^2+8xy+8y-8x+3y^2-4y+c. \qquad \ldots(b)$$

[from (α), (4) and (5)]

Now equating the two values of z from (a) and (b) when $y=2x+1$, the result is

$$c=-2,$$

or $$z+4x^2+y^2-8xy+8x-4y+2=0 \quad \text{[from (b)]}.$$

3. *Find a surface passing through the two lines $z=x=0$, $z-1=x-y=0$, satisfying $r-4s+4t=0$.* **(Agra 1963)**

Here $$r-4s+4t=0,$$

$$(D^2-4DD'+4D'^2)\,z=0,$$

$$(D-2D')^2=0.$$

$$\therefore \quad z=\phi\,(y+2x)+x\psi\,(y+2x). \qquad \ldots(1)$$

Now since the surface passes through

$$x=z=0 \text{ and } z-1=x-y=0,$$

$$\phi\,(2x+y)=0,\ x=0,\ z=0.$$

$$1=\phi\,(2x+y)+x\psi\,(2x+y),\ x-y=0.$$

$$\therefore \quad \psi\,(2x+y)=\frac{1}{x}=\frac{3}{3x}=\frac{3}{2x+x},\text{ and } x-y=0$$

$$=\frac{3}{(2x+y)}.$$

Also $\phi\,(2x+y)=0$.

$\therefore$ from (1), the required equation is

$$z=x\,\frac{3}{2x+y}$$

or $$z\,(2x+y)=3x.$$

4. *Find a surface satisfying*

$$2x^2r-5xys+2y^2t+2\,(px+qy)=0$$

and touching the hyperbolic paraboloid $z=x^2-y^2$ along its section by the plane $y=1$. **(Agra 1960)**

Here $$2x^2r-5xys+2y^2t+2\,(px+qy)=0$$

or $$2x^2\,\frac{\partial^2 z}{\partial x^2}-5xy\,\frac{\partial^2 z}{\partial x\,\partial y}+2y^2\,\frac{\partial^2 z}{\partial y^2}+2\left(x\,\frac{\partial z}{\partial x}+y\,\frac{\partial z}{\partial y}\right)=0.$$

Putting $x=e^u$, $y=e^v$ and denoting $\frac{\partial z}{\partial u}$ by D, $\frac{\partial z}{\partial v}$ by D', we get $[2D(D-1)-5DD'+2D'(D'-1)+2D+2D']\,z=0$

or $$(2D^2-5DD'+2D'^2)\,z=0$$

or $$(2D-D')(D-2D')=0.$$

$$\begin{aligned} z&=\phi_1(2v+u)+\phi_2(v+2u)\\ &=\phi_1(\log y^2x)+\phi_2(\log x^2y)\\ &=\psi_1(y^2x)+\psi_2(x^2y). \end{aligned} \qquad \ldots(F)$$

Now $$\frac{\partial z}{\partial x}=p=y^2\psi_1'(y^2x)+2xy\psi_2'(x^2y),$$

$$\frac{\partial z}{\partial y}=q=2xy\,\psi_1'(y^2x)+x^2\psi_2'(x^2y).$$

Also from $$z=x^2-y^2, \qquad \ldots(1)$$

$$\partial z/\partial x=2x,\ \partial z/\partial y=-2y.$$

Equating the values of p and q for points on $y=1$,

$$2yx\psi_2'(x^2y)+y^2\psi_1'(y^2x)=2x,$$

$$2yx\psi_1'(y^2x)+x^2\psi_2'(x^2y)=-2y.$$

From these two equations,

$$3yx^2\psi_2'(x^2y)=4x^2+2y^2$$

or $$-3xy^2\psi_1'(y^2x)=2x^2+4y^2.$$

Writing them as,

$$\psi_2'(x^2y)=\frac{4}{3y}+\frac{2y^2}{3x^2y}=\frac{4}{3}+\frac{2}{3x^2y}\ (\text{for } y=1),$$

$$\psi_1'(y^2x)=-\left[\frac{2}{3}\frac{x}{y^2}+\frac{4}{3x}\right]=-\left[\frac{2}{3}xy^2+\frac{4}{3xy^2}\right]$$

(for $y=1$).

$$\therefore\quad \psi_2(x^2y)=\tfrac{4}{3}x^2y+\tfrac{2}{3}\log(x^2y)+c_1 \qquad \ldots(2)$$

and $$\psi_1(y^2x)=-\frac{(xy^2)^2}{3}-\tfrac{4}{3}\log(xy^2)+c_2. \qquad \ldots(3)$$

$\therefore$ from (F), (2), and (3),

$$z=\tfrac{4}{3}x^2y-\frac{x^2y^4}{3}-\tfrac{6}{3}\log y+K. \qquad \ldots(4)$$

Now equating the values of z from (1) and (4) when $y=1$, we find that $K=-1$.

Hence from (4), the required equation is

$$3z=4x^2y-x^2y^4-6\log y-3.$$

Exercises 2 (3)

1. Find a surface satisfying $t=6x^3y$, containing the two lines, $y=0=z$, $y=1=z$. **Ans.** $z=x^3y^3+y(1-x^3)$.
2. Find a surface satisfying $s=8xy$ and passing through the circle $z=0=x^2+y^2-1$. **Ans.** $z=(x^2+y^2)^2-1$.
3. Find a surface satisfying $r=6x+2$ and touching $z=x^3+y^3$ along its section by the plane $x+y+1=0$. **Ans.** $z=x^3+y^3+(x+y+1)^2$.
4. Find a surface satisfying $r-2s+t=6$ and touching the hyperbolic paraboloid $z=xy$ along its section by the plane $y=x$. **Ans.** $z=x^2-xy+y^2$.

Spherical Harmonics

CHAPTER I

PART 1

Laplace's Equations in Rectangular and Polar Co-ordinates. Solid Spherical Harmonic of Degree n. Legendre's Equation. Legendre's Coefficients. Legendre's Polynomials. $P_n(x)$ and $Q_n(x)$.

1·1. $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}+\dfrac{\partial^2 V}{\partial z^2}=0$ is called Laplace's equation in rectangular co-ordinates. It is the very basis of Mathematical Physics. The polar form of Laplace's equation is

$$\frac{\partial^2 V}{\partial r^2}+\frac{2}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}=0,$$

where r, θ, ϕ are the polar co-ordinates of a point.

Any function $V_n=f_n(x, y, z)$ homogeneous in x, y, and z satisfying Laplace's equation is called a **Solid Spherical Harmonic** of degree n. Naturally V_n assumes the form $r^n Y_n(\theta, \phi)$, when put in polar co-ordinates, where $Y_n(\theta, \phi)$ is a function of θ and ϕ.

Now $Y_n(\theta, \phi)$ is called a **Surface Spherical Harmonic** of degree n. The suffix n is meaningful only in the sense that it indicates that $Y_n(\theta, \phi)$ is the coefficient of r^n in the process of putting $f_n(x, y, z)$ in polar co-ordinates, so that $V_n=f_n(x, y, z)=r^n Y_n(\theta, \phi)$.

1·11. Legendre's Equation. In the discussion of Surface Spherical Harmonics, we come across the equation $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$. This equation is known as Legendre's Equation.

1·12. To illustrate **the method of integration or solution of differential equations in series,** let us take the differential equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0. \qquad \ldots(A)$$

It can be solved or integrated in series of ascending or descending powers of x. **The solution in descending powers of x is more important than the one in ascending powers.** To find it out, let us take

$$y=\sum_{r=0}^{\infty} a_r x^{k-r} \qquad \ldots(1)$$

then
$$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r (k-r)\, x^{k-r-1},$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r (k-r)(k-r-1)\, x^{k-r-2}.$$

Substituting in the differential equation (A), we get

$$(1-x^2)\sum_{r=0}^{\infty} a_r (k-r)(k-r-1)\, x^{k-r-2}-2x\sum_{r=0}^{\infty} a_r (k-r)\, x^{k-r-1}$$
$$+n(n+1)\sum_{0}^{\infty} a_r x^{k-r}\equiv 0$$

or
$$\sum_{r=0}^{\infty}[(1-x^2)(k-r)(k-r-1)\, x^{k-r-2}-2(k-r)\, x^{k-r}$$
$$+n(n+1)\, x^{k-r}]a_r\equiv 0$$

or
$$\sum_{r=0}^{\infty}[(k-r)(k-r-1)\, x^{k-r-2}$$
$$+\{n(n+1)-(k-r)(k-r+1)\}x^{k-r}]\, a_r\equiv 0. \qquad \ldots(B)$$

Now (B) being an identity, we can equate to zero the coefficients of various powers of x.

In (B) equating to zero the coefficient of x^k which is the highest power of x, we have

$$a_0\{n(n+1)-k(k+1)\}=0.$$

Now $a_0 \neq 0$ as it is the coefficient of the very first term with which we begin to write the series (1) for y.

Hence $$\{n(n+1)-k(k+1)=0$$

or $$(n-k)(n+k+1)=0 \qquad \ldots(C)$$

This gives the value of k either as

$$k=n, \text{ or as } k=-n-1. \qquad \ldots(D)$$

Equating to zero the coefficient of the next lower power of x, that is of x^{k-1} in (B), we get

$$a_1\{n(n+1)-(k-1)k\}=0.$$

Now $\{n(n+1)-(k-1)k\} \neq 0$ by virtue of (D); hence

$$a_1=0. \qquad \ldots(E)$$

Equating to zero the coefficient of the general term namely that involving x^{k-r} in (B), we get

$$(k-r+2)(k-r+1)a_{r-2}+\{n(n+1)-(k-r)(k-r+1)\}a_r=0$$

or $$a_r=-\frac{(k-r+2)(k-r+1)}{(n-k+r)(n+k-r+1)}a_{r-2}. \qquad \ldots(F)$$

Results (D) and (F) are important. The former determines the index and the latter gives the relation between the coefficients a's. Combining both these,

$$\mathbf{a_r=-\frac{(n-r+2)(n-r+1)}{r.(2n-r+1)}a_{r-2}} \qquad \ldots(G)$$

when $\mathbf{k=n}$,

and $$\mathbf{a_7=\frac{(n+r-1)(n+r)}{(2n+1+r)(r)}a_{r-2}} \qquad \ldots(H)$$

when $\mathbf{k=-n-1}$.

Now, a_1 being zero from (E), we see from (F) that a_3, $a_5, \ldots, a_{2r+1}, \ldots$ are all zeros, and $a_2, a_4, \ldots a_{2r}, \ldots$ can be found out in terms of a_0 which is an arbitrary constant.

Taking $\mathbf{k=n}$ and accordingly the values of coefficients as

determined from (G) and substituting them in $y=\sum_{r=1}^{\infty} a_r x^{k-r}$, we get

$$y=a_0\left[x^n-\frac{n(n-1)}{2.(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}-\ldots\right] \quad \ldots(\alpha)$$

as one solution.

Taking $k=-n-1$ and accordingly the values of coefficients as determined from (H), we get

$$y=a_0\left[x^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5}+\ldots\right]\ldots(\beta)$$

as the other solution of our differential equation where a_0 in both cases is an arbitrary constant.

1·13. Definition of $P_n(x)$. **(Agra 1952, 53, 55)**

With n a positive integer and $a_0=\frac{1.3.5\ldots(2n-1)}{n!}$, the solution ($\alpha$) of Art. 1·12 is called $P_n(x)$.

In fact (as will be proved hereafter), it is that solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0,$$

which equals 1 when $x=1$.

$P_n(x)$, as seen from (α) of Art 1·12 is a terminating series and gives what are called **Legendre's Polynomials or coefficients,** *provided n is a positive integer.*

With n a positive integer and $a_0=\frac{n!}{1.3.5\ldots(2n+1)}$, the solution ($\beta$) of Art. 1·12, being a non-terminating series, is called $Q_n(x)$ and gives what are known as **Legendre's functions of the second kind for positive integral n.**

We have thus

$$P_n(x)=\frac{1.3.5\ldots(2n-1)}{n\,!}\left[x^n-\frac{n(n-1)}{2.(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}-\ldots\right],$$

$$Q_n(x)=\frac{n\,!}{1.3.5\ldots(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}x^{-n-5}+\ldots+\ldots\right].$$

The most general solution of the Legendre's equation is $y=AP_n(x)+BQ_n(x)$ where A and B are arbitrary constants.

In the next few chapter-parts we shall be more concerned with finding out various properties, various series and various integrals for $P_n(x)$.

Exercises 1 (1)

1. Integrate the equation $\frac{d^2y}{dx^2}+y=0$ in series.

Ans. $y=A\left(1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\ldots\right)+B\left(x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-\ldots\right)$

2. Show that

$$P_0(\mu)=1,\ P_1(\mu)=\mu,\ P_2(\mu)=\frac{3\mu^2-1}{2},\ P_3(\mu)=\frac{5\mu^3-3\mu}{2}.$$

3. Show that $P_n(-\mu)=(-1)^n P_n(\mu)$, $P_n(-1)=(-1)^n$.

4 By differentiating the series for $P_n(x)$, show that

$$\frac{d^mP_n(x)}{dx^m}=\frac{(2n)\,!}{2^n.n\,!\,(n-m)\,!}\left[x^{n-m}-\frac{(n-m)(n-m-1)}{2.(2n-1)}x^{n-m-2}+\frac{(n-m)(n-m-1)(n-m-2)(n-m-3)\,x^{n-m-4}}{2.4.(2n-1)(2n-3)}-\ldots\right].$$

CHAPTER I

PART 2

Generating Function for $P_n(\mu)$; $P_n(1)=1$; Rodrigue's Formula.

1·2. $P_n(\mu)$ is the coefficient of h^n in the expansion of

$$(1-2\mu h+h^2)^{-1/2}. \qquad \textbf{(Agra 1954)}$$

Now $(1-2\mu h+h^2)^{-1/2}=\{1-h\,(2\mu-h)\}^{-1/2}$

$$=1+\frac{1}{2}\,h\,(2\mu-h)+\frac{1.3}{2.4}\,h^2\,(2\mu-h)^2+\ldots$$

$$+\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n}h^n(2\mu-h)^n+\ldots$$

for sufficiently small value of h.

The coefficient of h^n in this expansion is

$$\frac{1.3.5\ldots 2n-1}{2.4.6\ldots 2n}\,(2\mu)^n-\frac{1.3.5\ldots 2n-3}{2.4\ldots 2n-2}.(2\mu)^{n-2}.{}^{n-1}C_1$$

$$+\frac{1.3.5\ldots 2n-5}{2.4.6\ldots 2n-4}\,(2\mu)^{n-4}.{}^{n-2}C_2$$

$$-\frac{1.3.5\ldots 2n-7}{2.4.6\ldots 2n-6}\,(2\mu)^{n-6}.{}^{n-3}C_3+\ldots$$

or

$$\frac{1.3.5\ldots 2n-1}{n\,!}\left\{\mu^n-\frac{n\,(n-1)}{2\,(2n-1)}\,\mu^{n-2}\right.$$

$$+\frac{n\,(n-1)(n-2)\,(n-3)\,\mu^{n-4}}{2^2.2\,!.(2n-1)\,(2n-3)}$$

$$\left.-\frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)}{2^3.3\,!.(2n-1)\,(2n-3)\,(2n-5)}\,\mu^{n-6}+\ldots\right\}$$

or $\dfrac{1.3.5\ldots(2n-1)}{n\,!}\Big\{\mu^n-\dfrac{n\,(n-1)}{2.(2n-1)}\,\mu^{n-2}$
$$+\frac{n\,(n-1)\,(n-2)\,(n-3)}{2.4.(2n-1)\,(2n-3)}\,\mu^{n-4}$$
$$-\frac{n\,(n-1)\,(n-2)\,(n-3)\,(n-4)\,(n-5)}{2.4.6.(2n-1)\,(2n-3)\,(2n-5)}\,\mu^{n-6}+\ldots\Big\}.$$

This is exactly $P_n(\mu)$.

Thus $\sum\limits_{n=0}^{\infty} \mathbf{h^n P_n}(\mu)=(\mathbf{1}-\mathbf{2}\mu\mathbf{h}+\mathbf{h^2})^{-1/2}$.

1·21. From the last article, $P_n(1)$ is the coefficient of h^n in the expansion of $\quad(1-2h+h^2)^{-1/2}$
or the coefficient of h^n in the expansion of $(1-h)^{-1}$
or in $\quad 1+h+h^2+\ldots+h^n+\ldots$

This coefficient is 1.

Hence $\quad \mathbf{P_n(1)=1.}$

This is a distinguishing property of Legendre's polynomials.

1·22. Rodrigue's Formula. **(Agra 1960, 1956)**

Let us form a differential equation in y, it being given that
$$y=(\mu^2-1)^n.$$

Now
$$\frac{dy}{d\mu}=n\,(\mu^2-1)^{n-1}\,(2\mu)$$

or
$$(\mu^2-1)\,\frac{dy}{d\mu}=2n\mu y. \qquad \ldots(1)$$

Differentiating (1) $(n+1)$ times by Leibnitz's theorem,
$$(\mu^2-1)\frac{d^{n+2}y}{d\mu^{n+2}}+{}^{n+1}C_1\,(2\mu)\,\frac{d^{n+1}y}{d\mu^{n+1}}+{}^{n+1}C_2.(2).\frac{d^ny}{d\mu^n}$$
$$=2n\left[\mu\,\frac{d^{n+1}y}{d\mu^{n+1}}+{}^{n+1}C_1\,\frac{d^ny}{d\mu^n}\right]$$

or
$$(\mu^2-1)\,\frac{d^{n+2}y}{d\mu^{n-2}}+2\mu\,\frac{d^{n+1}y}{d\mu^{n+1}}-n\,(n+1)\,\frac{d^ny}{d\mu^n}=0$$

or $$(\mu^2-1)\frac{d^2Z}{d\mu^2}+2\mu\frac{dZ}{d\mu}-n(n+1)Z=0, \qquad \ldots(2)$$

where $$Z=\frac{d^ny}{d\mu^n}=\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n.$$

Now (2) is Legendre's equation.

It is satisfied by Z and therefore by CZ, where C is a constant.

Thus CZ or $C\frac{d^ny}{d\mu^n}$ is a solution of Legendre's equation and this will be $P_n(\mu)$, if $C\frac{d^ny}{d\mu^n}=1$ when $\mu=1$ [Art. 1·21]

or if $$C\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n=1, \text{ when } \mu=1. \qquad \ldots(4)$$

In (4), considering $(\mu^2-1)^n$ as $(\mu-1)^n(\mu+1)^n$ and differentiating n times by Leibnitz's theorem and then putting $\mu=1$, we get $C.2^n.n!=1$, so that $C=\frac{1}{2^n.n!}$.

With this value of C, CZ or $C\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n$ is $P_n(\mu)$

or $$\frac{1}{2^n.n!}\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n=P_n(\mu).$$

This is **Rodrigue's Formula.**

Exercises 1 (2)

1. Integrate the **hypergeometric equation**

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{dy}{dx}-\alpha\beta y=0$$

in series of ascending powers of x, and show that the complete primitive is

$$AF(\alpha,\beta,\gamma,x)+Bx^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma.\ 2-\gamma,x),$$

where A and B are arbitrary constants and $F(\alpha,\beta,\gamma,x)$ stands for the series

$$1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2$$
$$+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots+\ldots$$

2. Show that $P_{2m+1}(0)=0$

and $$P_{2m}(0)=(-1)^m\frac{1.3.5\ldots 2m-1}{2.4.6\ldots 2m}.$$

3. Use Rodrigue's formula to show that $\int_{-1}^{1} P_n(\mu)\,d\mu=0$, except when $n=0$ in which case the integral is 2.

[It has to be remembered that

$$\left(\frac{d}{d\mu}\right)^{n-1}(\mu^2-1)^n=\left(\frac{d}{d\mu}\right)^{n-1}\{(\mu-1)^n(\mu+1)^n\}=0$$

for $\mu=1$ as well as for $\mu=-1$, because after evaluation by Leibnitz's theorem, we find that

$$\left(\frac{d}{d\mu}\right)^{n-1}\{(\mu-1)^n(\mu+1)^n\}$$

has both $(\mu-1)$ and $(\mu+1)$ as factors.]

4. $P_n(x)$ being defined as that solution of the Legendre's equation which equals 1 when $x=1$, use Rodrigue's formula to show that

$$P_n(x)=\frac{(2n)!}{2^n(n)!(n)!}\left\{x^n-\frac{n(n-1)x^{n-2}}{1.2.(2n-1)}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{2-4(2n-1)(2n-3)}x^{n-4}+\ldots\right\}.$$

Deduce from the above that $P_0(x)=1$, whatever x may be.

CHAPTER I

PART 3

Various trigonometrical series for $P_n(\mu)$

1·3. $P_n(\cos\theta)$ as a series in cosines of multiples of θ.

We know from Art. 2,

$$\sum_{n=0}^{\infty} h^n P_n(\cos\theta)=\{1-2(\cos\theta)h+h^2\}^{-1/2}$$

$$=\{1-(e^{i\theta}+e^{-i\theta})h+e^{i\theta}.e^{-i\theta}.h^2\}^{-1/2}.$$

$$=(1-e^{i\theta}h)^{-1/2}(1-e^{-i\theta}h)^{-1/2}. \qquad \ldots(1)$$

Now, $$(1-e^{i\theta}h)^{-1/2}=1+\tfrac{1}{2}e^{i\theta}h+\frac{1.3}{2.4}e^{2i\theta}h^2+\ldots$$

$$\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}e^{ni\theta}h^n+\ldots$$

$$(1-e^{-i\theta}h)^{-1/2}=1+\tfrac{1}{2}e^{-i\theta}h+\frac{1.3}{2.4}e^{-2i\theta}h^2+\ldots$$

$$\ldots+\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}e^{-ni\theta}h^n+\ldots$$

The coefficient of h^n in the product of the above two expansions on the right side of (1) is

$$\frac{1.3.5\ldots(2n-1)}{2.4.6.2n}\Big[(e^{in\theta}+e^{-in\theta})$$

$$+\frac{2n}{2n-1}(e^{i(n-2)\theta}+e^{-i(n-2)\theta}.\tfrac{1}{2}$$

$$+\frac{2n(2n-2)}{(2n-1)(2n-3)}(e^{i(n-4)\theta}+e^{-i(n-4)\theta})\frac{1.3}{2.4}$$

$$+\frac{2n(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}(e^{i(n-6)\theta}+e^{-i(n-6)\theta})\frac{1.3.5}{2.4.6}+\ldots\Big].$$

or $$\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\Big[2\cos(n\theta)+\frac{n}{2n-1}.2\cos(n-2)\theta$$
$$+\frac{n(n-1).1.3}{(2n-1)(2n-3).1.2}2.\cos(n-4)\theta$$
$$+\frac{n(n-1)(n-2)\,1.3.5}{(2n-2)(2n-3)(2n-5).1.2.3}.2\cos(n-6)\theta+\ldots\Big]$$

By equating coefficients of h^n on the two sides of (1), we have

$P_n(\cos\theta)$
$$=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots2n}\Big[2\cos n\theta+2.\frac{1.n}{1.(2n-1)}\cos(n-2)\theta$$
$$+2.\frac{1.3.n(n-1)}{1.2(2n-1)(2n-3)}\cos(n-4)\theta$$
$$+2.\frac{1.3.5}{1.2.3}\frac{n(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\cos(n-6)\theta+\ldots\Big].$$

1·31. $(1-2h\cos\theta+h^2)^{-1/2}=\{(1-h)^2+2h-2h\cos\theta\}^{-1/2}$
$$=\left\{(1-h)^2+4h\sin^2\frac{\theta}{2}\right\}^{-1/2}=\frac{1}{1-h}\left\{1-(-1)\frac{4h\sin^2\frac{\theta}{2}}{(1-h)^2}\right\}^{-\frac{1}{2}}$$
$$=\frac{1}{1-h}+\sum_{r=1}^{\infty}(-1)^r\frac{1.3.5\ldots(2r-1)}{2.4.6\ldots2r}.\frac{2^r.2^r.\sin^{2r}\frac{\theta}{2}h^r}{(1-h)^{2r+1}}.$$
$$=(1-h)^{-1}+\sum_{r=1}^{\infty}(-1)^r\frac{1.3.5\ldots(2r-1)}{2.4.6\ldots2r}2^r.2^r\sin^{2r}\frac{\theta}{2}$$
$$\times h^r(1-h)^{-2r-1}.$$

Equating the coefficients of h^n on the two sides of the above,

$$P_n(\cos\theta)=1+\sum_{r=1}^{\infty}(-1)^r\frac{1.3.5\ldots2r-1}{r!}.2^r\sin^{2r}\frac{\theta}{2}$$
$$\times\frac{(2r+1)(2r+2)\ldots(2r+1+n-r-1)}{(n-r)!}$$

$$=1+\sum_{r=1}^{\infty}(-1)^r.\frac{1.3.5\ldots 2r-1}{r!}.\frac{(n+r)!}{(2r)!\,(n-r)!}.2^r\sin^{2r}\frac{\theta}{2}.$$

$$=1+\sum_{r=1}^{\infty}(-1)^r.\frac{1}{r!.r!}(n-r+1)(n-r+2)\ldots(n+r)\sin^{2r}\frac{\theta}{2}.$$

$$=1-\frac{n(n+1)}{1!.1!}\sin^2\frac{\theta}{2}+\frac{(n-1)(n)(n+1)(n+2)}{2!.2!}\sin^4\frac{\theta}{2}$$

$$-\frac{(n-2)(n-1)(n)(n+1)(n+2)(n+3)}{3!.3!}\sin^6\frac{\theta}{2}+\ldots$$

$$=F\left(n+1,\,-n,\,1,\,\sin^2\frac{\theta}{2}\right).$$

Putting $\theta+\pi$ for θ, we see that

$$P_n(-\cos\theta)=F\left(n+1,\,-n,\,1,\,\sin^2\frac{\theta+\pi}{2}\right)$$

or,
$$(-1)^n P_n(\cos\theta)=F\left(n+1,\,-n,\,1,\,\cos^2\frac{\theta}{2}\right)$$

or,
$$P_n(\cos\theta)=(-1)^n F\left(n+1,\,-n,\,1,\,\cos^2\frac{\theta}{2}\right).$$

Exercises 1 (3)

1. Show that the Legendre's equation

$$(1-\mu^2)\frac{d^2y}{d\mu^2}-2\mu\frac{dy}{d\mu}+n(n+1)y=0$$

changes into the hypergeometric form

$$x(1-x)\frac{d^2y}{dx^2}+(\tfrac{1}{2}-\tfrac{3}{2}x)\frac{dy}{dx}+\frac{n(n+1)}{4}=0$$

by the transformation $\mu^2=x$.

Hence show by comparison that its complete primitive is

$$AF\left(-\frac{n}{2},\frac{n+1}{2},\frac{1}{2},\mu^2\right)+B\mu F\left(-\frac{n-1}{2},\frac{n+2}{2},\frac{3}{2},\mu^2\right).$$

2. Use Rodrigue's Formula to show that

$$P_n(x)=\frac{(-1)^n}{n!}\left(\frac{d}{dx}\right)^n\left\{(1-x)^n\left(1-\frac{1-x}{2}\right)^n\right\},$$

and then by expansion and carrying out of differentiations, show that

$$P_n(x)=F\left(-n, n+1, 1, \frac{1-x}{2}\right).$$

3. Show that $\int_0^\pi P_n(\cos\theta)\cos n\theta\, d\theta=\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}\,\pi.$

4. Show that $\int_0^\pi P_n(\cos\theta)\cos(n-1)\,\theta=0.$

5. Show that $\int_0^\pi P_n(\cos\theta)\cos m\theta\, d\theta=0$ if m and n are both positive integers such that $m+n$ is odd.

CHAPTER I

PART 4

Zeros of $P_n(\mu)$

1·4. *A value of μ for which a function $f(\mu)$ vanishes is said to be a zero of $f(\mu)$.*

A zero of $f(\mu)$ is a root of the equation $f(\mu)=0$.

1·401. Let $F(\mu)=(\mu-1)^n(\mu+1)^n$.

If we differentiate $F(\mu)$ by Leibnitz's theorem, we find that each of the differential coefficients $F'(\mu)$, $F''(\mu)$, $F'''(\mu),\ldots, F^{(n-2)}(\mu)$, $F^{(n-1)}(\mu)$ contains in every one of its terms, $(\mu-1)$ as well as $(\mu+1)$ as factors; hence *each of these differential coefficients of $F(\mu)$ ranging from the first to the $(n-1)$th will vanish for $\mu=1$ as well as for $\mu=-1$*, but the nth differential coefficient of $F(\mu)$ will have one term in which $(\mu-1)$ will not occur as a factor as well as one term in which $(\mu+1)$ will not occur as a factor. Hence *the nth differential coefficient of $F(\mu)$ will not vanish for either $\mu=1$.* or for $\mu=-1$.

1·402. *The zeros of $P_n(\mu)$, that is, the roots of the equation $P_n(\mu)=0$ are all real and lie between -1 and $+1$.*

Take a function $F(\mu)=(\mu-1)^n(\mu+1)^n$. We see that $F(\mu)$ vanishes for the end points $\mu=-1$ and $\mu=1$ and therefore by Rolle's theorem, $F'(\mu)$ vanishes for at least one intermediate value say μ_{11} of μ lying between -1 and $+1$, and obviously for the end points, $\mu=1$ and $\mu=-1$, and then by Rolle's theorem, $F''(\mu)$ vanishes in two intervals $(-1, \mu_{11})$ and $(\mu_{11}, 1)$, say, at least for $\mu=\mu_{21}$, and $\mu=\mu_{22}$ and also obviously for the end points $\mu=-1$ and $\mu=1$.

Applying Rolle's theorem again, we see that $F'''(\mu)$ vanishes in the three intervals $(-1, \mu_{21})$, (μ_{21}, μ_{22}), $(\mu_{22}, 1)$, say, at least for the values $\mu=\mu_{31}$, $\mu=\mu_{32}$, $\mu=\mu_{33}$ and also for the end points $\mu=-1$ and $\mu=1$ and therefore by the Rolle's theorem $F''''(\mu)$ vanishes in the four intervals $(-1, \mu_{31})$, (μ_{31}, μ_{32}), (μ_{32}, μ_{33}), $(\mu_{33}, 1)$, that is, it vanishes for four intermediate points at least as also for the end points $\mu=-1$ and $\mu=1$. Proceeding in this manner and applying Rolle's theorem successively, we find that

$$\left(\frac{d}{d\mu}\right)^n [(\mu-1)^n (\mu+1)^n], \quad \text{or} \quad \frac{1}{2^n.n\,!}\left(\frac{d}{d\mu}\right)^n (\mu^2-1)^n=P_n(\mu)$$

vanishes for n intermediate values $\mu_{n,1}, \mu_{n,2}, \ldots \mu_{n,n}$ of μ though not for the end values $\mu=-1$ and $\mu=1$. Thus, since $P_n(\mu)=0$ is an equation of the nth degree, it cannot have more than n roots. So, all the roots of $P_n(\mu)=0$ or all the zeros of $P_n(\mu)$ have now been obtained as lying between -1 and $+1$ on the real axis so that all the n roots are real and lie between $+1$ and -1.

1·41. All the roots of $\mathbf{P}_n(\mu)=0$ are different.

If they are not all different, at least two of them must be equal. Let their common value be α.

Then $P_n(\alpha)=0$, $P_n'(\alpha)=0$. $\quad [\because \;\; P_n(\mu)=(\mu-\alpha)^2\,\psi(\mu)]$

From the Legendre's equation,

$$(1-\mu^2)\frac{d^2P_n}{d\mu^2}-2\mu\frac{dP_n}{d\mu}+n(n+1)P_n=0.$$

$$P_n''(\alpha)=0$$

and from the Legendre's equation differentiated once,

$$P_n'''(\alpha)=0,$$

and then from the Legendre's equation differentiated twice,

$$P_n''''(\alpha)=0,$$

and so on.

Thus $P_n^{(n)}(\alpha)=0$ which is absurd as $P_n^{(n)}(\alpha)$ is always a non-zero constant namely $1.3.5\ldots(2n-1)$. Thus all the roots of $P_n(\mu)=0$ are different.

Let them be $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. Then if α is a root of $P_n(\mu)=0$, $-\alpha$ is also a root of $P_n(\mu)=0$ because of the relation $P_n(-\alpha)=(-1)^n P_n(\alpha)$. Therefore $-\alpha_1, -\alpha_2, -\alpha_3 \ldots, -\alpha_n$ are also roots of $P_n(\mu)$.

Thus

$P_n(\mu)=A(\mu^2-\alpha_1^2)(\mu^2-\alpha_2^2)\ldots(\mu^2-\alpha_{n/2}^2)$ if n is even.

$P_n(\mu)=B\mu(\mu^2-\beta_1^2)(\mu^2-\beta_2^2)\ldots(\mu^2-\beta^2_{(n-1)/2})$ if n is odd, A and B being certain constants.

Exercise 1 (4)

1. Show by the method of Integration in Series that the complete primitive of the hyper-geometric equation

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{dy}{dx}-\alpha\beta y=0$$

is
$$Ax^{-\alpha}F\left[\alpha, 1-\gamma+\alpha, 1-\beta+\alpha, \frac{1}{x}\right]$$
$$+Bx^{-\beta}F\left[\beta, 1-\gamma+\beta, 1-\alpha+\beta, \frac{1}{x}\right].$$

From the above deduce that the complete primitive of the Legendre's equation is

$$A\mu^n F\left(-\frac{n}{2}, \frac{1-n}{2}, \frac{1}{2}-n, \frac{1}{\mu^2}\right)$$
$$+B\mu^{-n-1}F\left(\frac{n+1}{2}, \frac{n+2}{2}, \frac{3}{2}+n, \frac{1}{\mu^2}\right).$$

[The first part will be $P_n(\mu)$ if n is a positive integer, for suitable value of the constant A. See Q. 1 of 1 (3).]

2. Do you agree that the following are incorrect expressions ?—

(a) 'The roots of a function.'

(b) 'The zeros of an equation.'

If so, please correct them.

3. n being any positive integer, show that all the zeros of $P_n(\mu)$ lie on concentric circles including a point circle in case n is odd. Also show that all these circles lie within and none without a certain circle concentric with them. Find the radius and centre of this circle.

CHAPTER I

PART 5

Laplace's definite integrals for $P_n(\mu)$.

1·5. We know from integral calculus,

$$\int_0^\pi \frac{d\phi}{a \pm b \cos\phi} = \frac{\pi}{\sqrt{(a^2-b^2)}}, \text{ where } a^2 > b^2.$$

Put $\qquad a = 1-\mu h,\ b = h\sqrt{(\mu^2-1)}$

and therefore $\qquad a^2-b^2 = 1-2\mu h+h^2.$

We get then,

$$\frac{\pi}{(1-2\mu h+h^2)^{1/2}} = \int_0^\pi \frac{d\phi}{1-h\mu \pm h\sqrt{(\mu^2-1)} \cos\phi}$$

$$= \int_0^\pi [1-h\{\mu \mp \sqrt{(\mu^2-1)} \cos\phi\}]^{-1}\, d\phi$$

$$= \int_0^\pi \left[\sum_{n=0}^{\infty} h^n \{\mu \mp \sqrt{(\mu^2-1)} \cos\phi\}^n\right] d\phi$$

for sufficiently small values of h

$$= \sum_{n=0}^{\infty} \int_0^\pi h^n \{\mu \mp \sqrt{(\mu^2-1)} \cos\phi\}^n\, d\phi. \quad \ldots(1)$$

Equating coefficients of h^n on the two sides of (1), we get

$$P_n(\mu) = \frac{1}{\pi} \int_0^\pi \{\mu\sqrt{\mp}(\mu^2-1) \cos\phi\}^n\, d\phi$$

or $$P_n(\cos\theta) = \frac{1}{\pi} \int_0^\pi (\cos\theta + i \sin\theta \cos\phi)^n\, d\phi,$$

where $\qquad \mu = \cos\theta.$ **(Agra 1953)**

This is the **first of the Laplace's integrals.**

1·51. $\displaystyle\int_0^{\pi} \frac{d\phi}{a \pm b \cos \phi} = \frac{\pi}{\sqrt{(a^2 - b^2)}}$ where $a^2 > b^2$. ...(1)

Put $a = \mu h - 1$, $b = h\sqrt{(\mu^2 - 1)}$; then

$$\frac{\pi}{\sqrt{(a^2 - b^2)}} = \frac{\pi}{\sqrt{(1 - 2\mu h + h^2)}} = \frac{\pi}{h\sqrt{\left(1 - \frac{2\mu}{h} + \frac{1}{h^2}\right)}}$$

$$= \frac{\pi}{h}\left(1 - \frac{2\mu}{h} + \frac{1}{h^2}\right)^{-1/2}$$

$$= \frac{\pi}{h} \cdot \sum_{n=0}^{\infty} \frac{1}{h^n} P_n(\mu) = \pi \sum_{n=0}^{\infty} \frac{1}{h^{n+1}} P_n(\mu). \quad \ldots(2)$$

Also $\displaystyle\int_0^{\pi} \frac{d\phi}{a \pm b \cos \phi} = \int_0^{\pi} \frac{d\phi}{h\{\mu \pm \sqrt{\{\mu^2 - 1)} \cos \phi\} - 1}$

$$= \int_0^{\pi} \frac{d\phi}{h\{\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi\}\left\{1 - \frac{1}{h\{\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi\}}\right\}}$$

$$= \int_0^{\pi} \left\{ \frac{\left(1 - \frac{1}{h\{\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi\}}\right)^{-1}}{h\{\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi\}} \right\} d\phi$$

$$= \int_0^{\pi} \left[\sum_0^{\infty} \frac{1}{h^{n+1}\{\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi\}^{n+1}} \right] d\phi$$

$$= \sum_{n=0}^{\infty} \frac{1}{h^{n+1}} \int_0^{\pi} \frac{d\phi}{\{\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi)^{n+1}}. \quad \ldots(3)$$

Hence from (1), (2) and (3)

$$\sum_{n=0}^{\infty} \frac{1}{h^{n+1}} P_n(\mu) = \sum_{n=0}^{\infty} \frac{1}{h^{n+1}} \int_0^{\pi} \frac{d\phi}{\{\mu \pm \sqrt{(\mu^2 - 1)} \cos \phi\}^{n+1}}. \quad \ldots(4)$$

Picking out and equating coefficients of $\frac{1}{h^{n+1}}$ on the two sides of (4), we have

$$\mathbf{P}_n(\mu) = \frac{\mathbf{1}}{\pi} \int_0^{\pi} \frac{\mathbf{d}\phi}{\{\mu \pm \sqrt{(\mu^2 - \mathbf{1})} \cos \phi\}^{n+1}}.$$

This is the **second of Laplace's integrals.**

From the two integrals of Laplace we easily see that

$$\mathbf{P}_n(\mu)=\mathbf{P}_{-n-1}(\mu).$$

Exercise 1 (5)

1. The following is due to Euler :

$$P_n(\cos\theta)=\cos^n\theta\, F\left(-\frac{n}{2}, \frac{1}{2}-\frac{n}{2}, 1, -\tan^2\theta\right).$$

[In the integral of Art. 1·5, put

$(\cos\theta+i\sin\theta\cos\phi)^n$ as $\cos^n\theta\,(1+\tan\theta\cos\phi)^n$.

Expand and integrate, thus getting the result.]

2. Show that

$$P_n(\cos\theta)=\cos^{2n}\frac{\theta}{2}\, F\left(-n, -n, 1, -\tan^2\frac{\theta}{2}\right).$$

(Murphy)

[In the first of Laplace's integrals for $P_n(\cos\theta)$, namely

$$P_n(\cos\theta)=\frac{1}{\pi}\int_0^{\pi}(\cos\theta+i\sin\theta\cos\phi)^n\,d\phi,$$

put $(\cos\theta+i\sin\theta\cos\phi)^n$ in factors which is then

$$\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}e^{i\varphi}\right)^n\times\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}e^{-i\varphi}\right)^n$$

or $$\cos^{2n}\frac{\theta}{2}\left(1+i\tan\frac{\theta}{2}e^{i\varphi}\right)^n\times\left(1+i\tan\frac{\theta}{2}e^{-i\varphi}\right)^n.$$

Expanding, multiplying, and integrating, we get the desired result.]

CHAPTER I

PART 6

Recurrence formulae

1·6. Let $V=(1-2\mu h+h^2)^{-1/2}=\sum\limits_{n=0}^{\infty} h^n P_n(\mu)$; ...(1)

then from (1), $V^2(1-2\mu h+h^2)\equiv 1$; $\dfrac{dV}{dh}\equiv\sum\limits_{n=0}^{\infty} nh^{n-1}P_n(\mu)$.

Differentiating $V^2(1-2\mu h+h^2)\equiv 1$ with respect to h and dividing by $2V$, we have

$$V(h-\mu)+\frac{dV}{dh}(1-2\mu h+h^2)\equiv 0$$

or $(h-\mu)\,\Sigma\, h^n P_n(\mu)+(1-2\mu h+h^2).\Sigma\, nh^{n-1}P_n(\mu)\equiv 0$. ...(2)

Equating to zero the coefficient of h^{n-1} in (2), we get

$$P_{n-2}-\mu P_{n-1}+nP_n-2\mu(n-1)P_{n-1}+(n-2)P_{n-2}=0$$

or
$$\mathbf{nP_n=(2n-1)\,\mu P_{n-1}-(n-1)\,P_{n-2}}\,. \qquad \text{...(I)}$$

(Agra 1961, 1953)

1·61. To prove

$$(\mu^2-1)\frac{\mathbf{dP}_n}{\mathbf{d}\mu}=\mathbf{n}(\mu\mathbf{P}_n-\mathbf{P}_{n-1})=-(\mathbf{n}+1)(\mu\mathbf{P}_n-\mathbf{P}_{n+1}).$$

Now, μP_n-P_{n-1}

$$=\frac{1}{\pi}\mu\int_0^{\pi}[\mu+\sqrt{(\mu^2-1)}\cos\phi]^n\,d\phi$$

$$-\frac{1}{\pi}\int_0^{\pi}[\mu+\sqrt{(\mu^2-1)}\cos\phi]^{n-1}\,d\phi$$

$$=\frac{1}{\pi}\int_0^{\pi}\Big[\{\mu+\sqrt{(\mu^2-1)}\cos\phi\}^{n-1}$$

$$\times\{\mu^2+\mu\sqrt{(\mu^2-1)}\cos\phi-1\}\Big]\,d\phi$$

$$=\frac{\mu^2-1}{\pi}\int_0^\pi \{\mu+\sqrt{(\mu^2-1)}\cos\phi\}^{n-1}\left(1+\frac{\mu\cos\phi}{\sqrt{(\mu^2-1)}}\right)d\phi$$

$$=\frac{(\mu^2-1)}{n\pi}\frac{d}{d\mu}\left[\int_0^\pi \{\mu+\sqrt{(\mu^2-1)}\cos\phi\}^n\,d\phi\right]$$

$$=\frac{\mu^2-1}{n}\frac{dP_n}{d\mu}. \quad \text{[Art. 1·5]}$$

Hence

$$\left.\begin{aligned}&(\mu^2-1)\frac{\mathbf{dP}_n}{\mathbf{d}\mu}=\mathbf{n}\,(\mu\mathbf{P}_n-\mathbf{P}_{n-1}). &&\ldots(\mathrm{A})\\ & &&\textbf{(Agra 1959)}\\ &(\mu^2-1)\frac{\mathbf{dP}_n}{\mathbf{d}\mu}=-(\mathbf{n}+\mathbf{1})\,(\mu\mathbf{P}_n-\mathbf{P}_{n+1}). &&\ldots(\mathrm{B})\\ & &&\textbf{(Agra 1957)}\end{aligned}\right\}\ldots(\mathrm{II})$$

We get (B) from (A) by putting $-n-1$ for n and remembering that $P_{-n-1}=P_n$, $P_{-n-2}=P_{n+2-1}=P_{n+1}$.

1·62. We have $(\mu^2-1)\dfrac{d^2P_n}{d\mu^2}+2\mu\dfrac{dP_n}{d\mu}-n\,(n+1)\,P_n=0$

[Legendre's equation]

or $$\frac{d}{d\mu}\left\{(\mu^2-1)\frac{dP_n}{d\mu}\right\}=n\,(n+1)\,P_n$$

or $$\frac{d}{d\mu}\left[n\,(\mu P_n-P_{n-1})\right]=n\,(n+1)\,P_n \text{ [using (II), (A)]}$$

or $$\frac{d}{d\mu}(\mu P_n-P_{n-1})=(n+1)\,P_n$$

or $$\mu\frac{\mathbf{dP}_n}{\mathbf{d}\mu}-\frac{\mathbf{dP}_{n-1}}{\mathbf{d}\mu}=\mathbf{nP}_n. \quad \ldots(\mathrm{III})$$

(Agra 1954)

Also, $$-\mu\frac{\mathbf{dP}_n}{\mathbf{d}\mu}+\frac{\mathbf{dP}_{n+1}}{\mathbf{d}\mu}=(\mathbf{n}+\mathbf{1})\,\mathbf{P}_n \quad \ldots(\mathrm{IV})$$

[putting $-n-1$ for n in (III) and remembering that

$P_{-n-1}=P_n$, $P_{-n-2}=P_{n+2-1}=P_{n+1}$].

By adding (III) and (IV), we get

$$(2n+1)\ P_n=\frac{dP_{n+1}}{d\mu}-\frac{dP_{n-1}}{d\mu}. \qquad \ldots(V)$$

(Agra 1961)

1·63. By putting $n-1$ for n in (V),

$$\frac{dP_n}{d\mu}=(2n-1)\ P_{n-1}+\frac{dP_{n-2}}{d\mu} \qquad \ldots(A)$$

(Agra 1955, 52)

$$=(2n-1)\ P_{n-1}+(2n-5)\ P_{n-3}+\frac{dP_{n+4}}{d\mu}$$

$$\left[\text{as, by putting } n-2 \text{ for } n \text{ in (A), we get } \frac{dP_{n-2}}{d\mu}=(2n-5)\ P_{n-3}+\frac{dP_{n-4}}{d\mu}\right].$$

Thus by repeated application of (A), we get

$$\frac{dP_n}{d\mu}=(2n-1)\ P_{n-1}+(2n-5)\ P_{n-3}+(2n-9)\ P_{n-5}+\ldots \qquad \ldots(VI)$$

(Agra 1956)

the last term being $3P_1$ or P_0 according as n is even or odd.

This is called **Christoffel's expansion.**

1·64. Multiplying (A) and (B) of (II) respectively by $n+1$ and n, we have

$$(n+1)\ (\mu^2-1)\ \frac{dP_n}{d\mu}=n\ (n+1)\ (\mu P_n-P_{n-1}),$$

$$n\ (\mu^2-1)\ \frac{dP_n}{d\mu}=-(n+1)\ n.(\mu P_n-P_{n+1}).$$

Adding these, we get

$$(2n+1)(\mu^2-1)\ \frac{dP_n}{d\mu}=n(n+1)\ (P_{n+1}-P_{n-1})$$

or

$$(\mu^2-1)\ \frac{dP_n}{d\mu}=\frac{n\ (n+1)}{2n+1}\ (P_{n+1}-P_{n-1}). \qquad \ldots(VII)$$

This is called **Beltrami's result.**

Exercises 1 (6)

1. Show that

$$\frac{dP_n}{dx}-\frac{dP_{n-2}}{dx}=(2n-1)\,P_{n-1}.$$ **(Agra 55, 52)**

2. Show that
$\mu P'_n(\mu)=nP_n(\mu)+(2n-3)\,P_{n-2}(\mu)+(2n-7)\,P_{n-4}(\mu)+\ldots$
(Agra Final 1959)

3. Show that $xP'_{10}-P'_9=10P_{10}$.

4. Show that $11\,(x^2-1)\,P_5'=30\,(P_6-P_4)$.

PART 7

Christoffel's Summation Formula for the sum of the first $n+1$ terms of the series $\sum_{r=0}^{n} (2r+1)\, P_r(x)\, P_r(y)$.

Orthogonal Properties of $P_n(\mu)$.

1·7. Now, from recurrence formula (1) of Art. 1·6,

$$(2r+1)\, x.P_r(x)\, P_r(y) = P_r(y)\, \{(r+1)\, P_{r+1}(x) + rP_{r-1}(x)\}.$$

$$(2r+1)\, yP_r(y)\, P_r(x) = P_r(x)\, \{(r+1)\, P_{r+1}(y) + rP_{r-1}(y)\}.$$

Subracting, $\sum_{r=0}^{n} (2r+1)\, P_r(x)\, P_r(y).(x-y)$

$$=\sum_{r=0}^{n} (r+1)\, [P_{r+1}(x)\, P_r(y) - P_{r+1}(y)\, P_r(x)$$

$$-r\, \{P_r(x)\, P_{r-1}(y) - P_r(y)\, P_{r-1}(x)\}]$$

$$=(n+1)\, \{P_{n+1}(x)\, P_n(y) - P_{n+1}(y)\, P_n(x)\}.$$

Hence $\sum_{r=0}^{n} (2r+1)\, P_r(x)\, P_r(y)$

$$=(n+1)\, \frac{P_{n+1}(x)\, P_n(y) - P_{n+1}(y)\, P_n(x)}{x-y}.$$

This is **Christoffel's Summation Formula.**

(Agra 1958)

1·71. (a) $\int_{-1}^{1} P_m(\mu)\, P_n(\mu)\, d\mu = 0$ (n and m are positive integers).

(Agra 1955, 52)

(b) $\int_{-1}^{1} P_n^2(\mu)\, d\mu = \frac{2}{2n+1}$ (n is a positive integer).

(Agra 1962, 55)

(a) $\int_{-1}^{1} P_n(\mu)\, P_m(\mu)\, d\mu$

$$=\frac{1}{2^{m+n}m\,!.n\,!}\int_{-1}^{1}\left(\frac{d}{d\mu}\right)^n (\mu^2-1)^n.\left(\frac{d}{d\mu}\right)^m (\mu^2-1)^m\, d\mu$$

$$=K\left[\left(\frac{d}{d\mu}\right)^m (\mu^2-1)^m.\left(\frac{d}{d\mu}\right)^{n-1} (\mu^2-1)^n\right]_{-1}^{1}$$

$$-K\int_{-1}^{1}\left(\frac{d}{d\mu}\right)^{m+1} (\mu^2-1)^m.\left(\frac{d}{d\mu}\right)^{n-1} (\mu^2-1)^n\, d\mu,$$

where $K=\frac{1}{2^{m+n}}\frac{1}{n\,!}.\frac{1}{m\,!}$, and $n > m$.

The first portion vanishes at both the limits. We integrate the second portion by parts again and continue the process till we get the integral to be

$$(-1)^m.K\int_{-1}^{1}\frac{d^{n-m}(\mu^2-1)^n}{d\mu^{n-m}}.\frac{d^{2m}(\mu^2-1)^m}{d\mu^{2m}}\,d\mu$$

$$=(-1)^m K.(2m)\,!\int_{-1}^{1}\frac{d^{n-m}(\mu^2-1)^n}{d\mu^{n-m}} \qquad \ldots(1)$$

$$=(-1)^m.K.(2m)\,!\left[\frac{d^{n-m-1}(\mu^2-1)^n}{d\mu^{n-m-1}}\right]_{-1}^{1}=0$$

[as the expression within brackets vanishes for both limits.]

(b) From (1), if $m=n$, the integral becomes

$$\int_{-1}^{1} P_n^2(\mu)\, d\mu=(-1)^n.\frac{(2n\,!)}{2^n.2^n.n\,!.n\,!}\int_{-1}^{1}(\mu^2-1)^n\, d\mu$$

$$=\frac{1.3.5\ldots(2n-1)}{2^n.n\,!}\int_{-1}^{1}(1-\mu^2)^n\, d\mu$$

$$=\frac{1.3.5\ldots(2n-1)}{2^n.n\,!}\int_{0}^{\pi}\sin^{2n+1}\theta\, d\theta$$

(where $\mu=\cos\theta$)

$$=2.\frac{1.3.5\ldots(2n-1)}{2^n.n\,!}\int_{0}^{\pi/2}\sin^{2n+1}\theta\, d\theta$$

$$=2.\frac{1.3.5...(2n-1)}{2^n.n!}.\frac{2n(2n-2)...2}{(2n+1)(2n-1)...3}$$

[by Wallis's formula]

$$=\frac{2}{2n+1}.$$

1·72. Solved Examples.

Example 1. *Prove that*

$$\frac{1-z^2}{(1-2\mu z+z^2)^{3/2}}=\sum_{n=0}^{\infty}(2n+1)\,z^n P_n(\mu). \quad \textbf{(Agra 1959)}$$

$$\frac{1-z^2}{(1-2\mu z+z^2)^{3/2}}=\frac{(1-2\mu z+z^2)}{(1-2\mu z+z^2)^{3/2}}+\frac{2\mu z-2z^2}{(1-2\mu z+z^2)^{3/2}}$$

$$=\frac{1}{\sqrt{(1-2\mu z+z^2)^{1/2}}}+2z\left\{-\frac{(-\mu+z)}{(1-2\mu z+z^2)^{3/2}}\right\}$$

$$=\frac{1}{\sqrt{(1-2\mu z+z^2)^{1/2}}}+2z\,\frac{d}{dz}\left\{\frac{1}{\sqrt{(1-2\mu z+z^2)^{1/2}}}\right\}$$

$$=\sum_{n=0}^{\infty}z^n P_n(\mu)+2z.\frac{d}{dz}\left\{\sum_{n=0}^{\infty}z^n P_n(\mu)\right\}$$

[Art. 1·2]

$$=\sum_{n=0}^{\infty}z^n P_n(\mu)+2z.\sum_{n=0}^{\infty}nz^{n-1}P_n(\mu)$$

$$=\sum_{n=0}^{\infty}z^n P_n(\mu)+\sum_{n=0}^{\infty}2nz^n P_n(\mu)$$

$$=\sum_{n=0}^{\infty}(2n+1)\,z^n P_n(\mu).$$

Example 2. *Show that*

$$\frac{1+z}{z(1-2xz-z^2)^{1/2}}-\frac{1}{z}=\sum_{n=0}^{\infty}\{P_n(x)+P_{n+1}(x)\}\,z^n.$$

(Agra 1963)

Left hand side

$$=\frac{1}{z}.\frac{1}{(1-2xz-z^2)^{1/2}}+\frac{1}{(1-2xz+z^2)^{1/2}}-\frac{1}{z}.$$

$$=\frac{1}{z}\sum_{n=0}^{\infty} P_n(x)\, z^n+\sum_{n=0}^{\infty} P_n(x)\, z^n-\frac{1}{z}$$

$$=\frac{1}{z}+\sum_{n=1}^{\infty} \frac{P_n(x)\, z^n}{z}+\sum_{n=0}^{\infty} P_n(x)\, z^n-\frac{1}{z}$$

$$=\sum_{n=1}^{\infty} P_n(x)\, z^{n-1}+\sum_{n=0}^{\infty} P_n(x)\, z^n$$

$$=\sum_{n=0}^{\infty} P_{n+1}(x)\, z^n+\sum_{n=0}^{\infty} P_n(x)\, z^n$$

$$=\sum_{n=0}^{\infty} \{P_{n+1}(x)+P_n(x)\}\, z^n.$$

Example 3. *Show that* $C+\int P_n\, dx=\dfrac{P_{n+1}-P_{n-1}}{2n+1}$.

(Agra 1957)

Proceed as in Art. 1·62 and get the result,

$$(2n+1)\, P_n(x)=\frac{dP_{n+1}(x)}{dx}-\frac{dP_{n-1}(x)}{dx}$$

or

$$P_n(x)=\frac{1}{2n+1}\{P'_{n+1}(x)-P'_{n-1}(x)\}.$$

$$\therefore \quad \int P_n(x)\, dx+C=\frac{P_{n+1}(x)-P_{n-1}(x)}{2n+1}.$$

Example 4. *Show that* $\dfrac{dP_n}{dx}-\dfrac{dP_{n-2}}{dx}=(2n-1)\, P_{n-1}$.

(Agra 1952, 1955)

Procceed as in Art. 1·62 and get the result

$$(2n+1)\, P_n=\frac{dP_{n+1}}{dx}-\frac{dP_{n-1}}{dx}.$$

Putting $n-1$ for n in the above, we have

$$(2n-1)\, P_{n-1}=\frac{dP_n}{dx}-\frac{dP_{n-2}}{dx}.$$

Example 5. *Show that* $\int_{-1}^{1}\left(\frac{dP_n}{d\mu}\right)^2 d\mu = n(n+1).$

(Agra 1956)

Christoffel's expansion for $\frac{dP_n}{d\mu}$ as given in result (VI) in Art. 1·63 is

$$\frac{dP_n}{d\mu}=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}+\ldots$$
$$\ldots+1\,.\,P_0 \text{ if } n \text{ is odd.} \quad \ldots(A)$$
$$\frac{dP_n}{d\mu}=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+(2n-9)\,P_{n-5}+\ldots$$
$$\ldots+3P_1 \text{ if } n \text{ is even.} \quad \ldots(B)$$

If a is the first term of an Arithmetic Series and b is the N^{th} term and d the common difference, then

$$b=a+(N-1)\,d$$

or

$$\frac{b-a}{d}+1=N. \qquad \ldots(1)$$

Hence, the number of terms in (A)

$$=\frac{(2n-1)-1}{4}+1=\frac{n+1}{2} \qquad \ldots(2)$$

and the number of terms in (B)

$$=\frac{(2n-1)-3}{4}+1=\frac{n}{2}. \qquad \ldots(3)$$

Squaring and integrating (A) between the limits indicated,

$$\int_{-1}^{1}\left(\frac{dP_n}{d\mu}\right)^2 d\mu$$
$$=(2n-1)^2\int_{-1}^{1}P^2_{n-1}\,d\mu+(2n-5)^2\int P^2_{n-3}\,d\mu+\ldots+1^2\int_{-1}^{1}P_0^2\,d\mu$$
$$+(2n-1)\,(2n-5)\int_{-1}^{1}P_{n-1}P_{n-3}\,d\mu+(2n-1)(2n-9)\int_{-1}^{1}P_{n-1}P_{n-5}\,d\mu$$
$$+\ldots+\ldots+\ldots+5\,.\,1\int_{-1}^{1}P_2P_0\,d\mu.$$

$$=(2n-1)^2.\frac{2}{2\ (n-1)+1}+\frac{(2n-5)^2.2}{2\ (n-3)+1}+\ldots 1^2.\frac{2}{2.0+1}+0+0+\ldots+0\ldots+\ldots 0$$

[Art. 1·71, (a) and (b)]

$$=2\ [(2n-1)+(2n-5)+\ldots+1]$$

$$=2\left[\frac{n+1}{4}\ (2n-1+1)\right]=\mathbf{n\ (n+1)}. \qquad \ldots(\mathrm{F_1})$$

The number of terms added above is $\dfrac{\overline{2n-1}-1}{4}+1$

$$=\frac{2n+2}{4}=\frac{n+1}{2} \quad \text{[by (1)]}.$$

Similarly squaring and integrating (B),

$$\int_{-1}^{1}\left(\frac{dP_n}{d\mu}\right)^2 d\mu$$

$$=(2n-1)^2.\frac{2}{2\ (n-1)+1}+\frac{(2n-5)^2.2}{2\ (n-3)+1}+\ldots 3^2.\frac{2}{2.1+1}$$

+ integrals which vanish between the limits

$$=2\ [(2n-1)+(2n-5)+\ldots+3]$$

$$=2\left[\frac{n}{4}\ (2n-1+3)\right]=\mathbf{n\ (n+1)}. \qquad \ldots(\mathrm{F_2})$$

The number of terms added is $\dfrac{2n-1-3}{4}+1=\dfrac{n}{2}$ [by (1)].

(F_1) and (F_2) prove the result.

Example 6. *Show that*

$P_0+3P_1+5P_2+\ldots+(2n-1)\ P_n=P'_{n+1}+P'_n$. **(Agra 1963)**

From recurrence formula (V) of Art. 1·62, we have

$$(2n+1)\ P_n=P'_{n+1}-P'_{n-1}. \qquad \ldots(1)$$

Giving to n values 1, 2, 3, ...n, we get

$$3P_1=P_2'-P_0',$$
$$5P_2=P_3'-P_1',$$
$$7P_3=P_4'-P_2',$$
$$9P_4=P_5'-P_3',$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$(2n+1)\ P_n=P'_{n+1}-P'_{n-1}.$$

Adding, we get

$$3P_1+5P_2+\ldots(2n+1)\,P_n=P'_{n+1}+P_n'-P_1'-P_0'$$
$$=P'_{n+1}+P_n'-1$$
$$[\because\ P_1'=1,\ P_0'=0]$$

or $\mathbf{P_0+3P_1+5P_2+\ldots+(2n+1)\,P_n=P'_{n+1}+P_n'}$ $[\because\ P_0=1]$.

It has to be remembered that $P_1(x)=x$, $P_0(x)=1$ as is evident from Rodrigue's formula

Example 7. *Show that*

$$\int_{-1}^{1}(1-x^2)\left(\frac{dP_n}{dx}\right)^2 dx=\frac{2n\,(n+1)}{2n+1}.$$ **(Agra Final 1945)**

Results (VII) and (VI) of Articles 1·64 and 1·63 are respectively as follows :

$$(1-x^2)\,\frac{dP_n}{dx}=\frac{n\,(n+1)}{2n+1}\,(P_{n-1}-P_{n+1}), \qquad \ldots(1)$$

$$\frac{dP_n}{dx}=(2n-1)\,P_{n-1}+(2n-5)\,P_{n-3}+\ldots \qquad \ldots(2)$$

Multiplying (1) and (2), and integrating

$$\int_{-1}^{1}(1-x^2)\left(\frac{dP_n}{dx}\right)^2 dx$$

$$=\frac{n\,(n+1)}{2n+1}.(2n-1)\int_{-1}^{1}P^2_{n-1}\,dx+\text{integrals of the}$$

products of two different P's, which vanish

$$=\frac{n\,(n+1)}{2n+1}.(2n-1)\,\frac{2}{2\,(n-1)+1}$$

$$=\frac{2n\,(n+1)}{2n+1}.$$

Example 8. *Show that*

$$P_0^2+3P_1^2+5P_2^2+\ldots+(2n+1)\,P_n^2$$
$$=(n+1)\,(P_nP'_{n+1}-P_{n+1}\,P_n').$$

Now from Christoffel's summation formula,

$$\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(y)$$

$$=(n+1).\frac{P_{n+1}(x)\,P_n(y)-P_{n+1}(y)\,P_n(x)}{x-y}.$$

Putting $y=x+\epsilon$, where ϵ is an arbitrarily small quantity, we get

$$\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(x+\epsilon)$$

$$=(n+1)\,\frac{P_{n+1}(x)\,P_n(x+\epsilon)-P_{n+1}(x+\epsilon)\,P_n(x)}{x-(x+\epsilon)}$$

$$=(n+1)\frac{P_{n+1}(x)\left\{P_n(x)+\epsilon P_n'(x)+\frac{\epsilon^2}{2\,!}P_n''(x)+\ldots\right\}-\left\{P_{n+1}(x)+\epsilon P'_{n+1}(x)+\ldots\right\}P_n(x)}{-\epsilon}$$

[by expansion by Taylor's theorem].

Taking limits as $\epsilon\to 0$,

$$P_0^2+3P_1^2+5P_2^2+(2n+1)P_n^2$$
$$=(n+1)\,\{P'_{n+1}(x)\,P_n(x)-P_{n+1}(x)\,P_n'(x)\}. \quad \ldots\text{(F)}$$

Example 9. *Show that*

$$(n+1)^2\,P_n^2-(x^2-1)\,P_n'^2=(n+1)\,(P_nP'_{n+1}-P_{n+1}P_n').$$

From (II) (B) of Art 1·61, we have

$$(x^2-1)\,P_n'=-(n+1)\,(xP_n-P_{n+1}).$$

Multiplying the above by P_n', we get

$$-(n+1)\,P_n'\,(xP_n-P_{n+1})=(x^2-1)\,P_n'^2. \quad \ldots(1)$$

Again from result (IV) of art. 1·62, we have

$$-xP_n'+P'_{n+1}=(n+1)\,P_n.$$

Multiplying the above by $(n+1)\,P_n$, we get

$$(n+1)\,P_n\,(-xP_n'+P'_{n+1})=(n+1)^2\,P_n^2. \quad \ldots(2)$$

Subtracting (1) from (2), we have

$$(n+1)\,[P_nP'_{n+1}-P_{n+1}P_n']=(n+1)^2\,(P_n)^2-(x^2-1)\,(P_n')^2\ldots\text{(F)}$$

Example 10. *Show that*

$$P_0^2+3P_1^2+5P_2^2+\ldots+(2n+1)\,P_n^2$$
$$=(n+1)^2(P_n)^2-(x^2-1)\,(P_n')^2. \qquad \textbf{(Agra 58)}$$

Proceed as in Example 8 and get the result (F) of that example.

Proceed as in Example 9 and get the result (F) of that example.

From the two results (F) of these two examples, we have

$$P_0^2+3P_1^2+5P_2^2+\ldots+(2n+1)\,P_n^2$$
$$=(n+1)^2\,(P_n)^2-(x^2-1)\,(P_n')^2.$$

Ex. 11. *Prove that*

$$\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\,dx=\frac{2n\,(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}.$$

(Agra 1962, Final 1959)

From recurrence formula 1 of Art. 1·6, we have

$$(2n-1)\,xP_{n-1}=nP_n+(n-1)\,P_{n-2}. \qquad \ldots(1)$$

Putting $(n+2)$ for n in (1), we have

$$(2n+3)xP_{n+1}=(n+2)\,P_{n+2}+(n+1)\,P_n. \qquad \ldots(2)$$

Multiplying (1) and (2), and integrating,

$$(2n-1)\,(2n+3)\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\,dx$$

$=n\,(n+1)\int_{-1}^{1} P_n^2\,dx+$integrals of products of different P's, which vanish.

Hence $$\int_{-1}^{1} x^2 P_{n+1} P_{n-1}\,dx=\frac{n\,(n+1)}{(2n-1)\,(2n+3)}\cdot\frac{2}{2n+1}$$
$$=\frac{2n\,(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}.$$

Ex. 12. *Prove that* $(1-2xz+z^2)^{-1/2}$ *is a solution of the equation*

$$\frac{z\,\partial^2(zv)}{\partial z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\,\frac{\partial v}{\partial x}\right\}=0.$$

(Agra 1957)

Now $\quad (1-2xz+z^2)^{-1/2}=\sum\limits_{0}^{\infty} z^n P_n(x)$

Substituting $\sum\limits_{n=0}^{\infty} z^n P_n(x)$ for v, we have

$$z\,\frac{\partial^2 (zv)}{\partial z^2}=z.\left(\frac{\partial}{\partial z}\right)^2\left\{z.\sum_{n=0}^{\infty} z^n P_n(x)\right\}$$

$$=z.\left(\frac{\partial}{\partial z}\right)^2\left\{\sum_{n=0}^{\infty} z^{n+1} P_n(x)\right\}$$

$$=z\left\{\sum_{n=0}^{\infty}(n+1)\,nz^{n-1}P_n(x)\right\}$$

$$=\sum_{n=0}^{\infty} n\,(n+1)\,z^n P_n(x)$$

$$=\sum_{n=0}^{\infty}\{n\,(n+1)\,P_n(x)\}\,z^n. \qquad \ldots(1)$$

$$\frac{\partial}{\partial x}\left\{(1-x^2)\,\frac{\partial v}{\partial x}\right\}=\frac{\partial}{\partial x}\left\{(1-x^2).\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} z^n P_n(x)\right)\right\}$$

$$=(1-x^2)\left(\frac{\partial}{\partial x}\right)^2\left\{\sum_{n=0}^{\infty} z^n P_n(x)\right\}-2x\frac{\partial}{\partial x}\left\{\sum_{n=0}^{\infty} z^n P_n(x)\right\}$$

$$=(1-x^2)\sum_{n=0}^{\infty} z^n P''_n(x)-2x\sum_{n=0}^{\infty} z^n P_n'(x)$$

$$=\sum_{n=0}^{\infty}(1-x^2)\,P_n''(x)\,z^n-\sum_{n=0}^{\infty} 2x\,P_n'(x)\,z^n. \qquad \ldots(2)$$

Hence

$$z\frac{\partial^2 (zv)}{\partial z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\}=\sum_{n=0}^{\infty} n(n+1)\,P_n(x)\,z^n$$

$$+\sum_{n=0}^{\infty}(1-x^2)\,P_n''(x)\,z^n-\sum_{n=0}^{\infty}2xP_n'(x)\,z^n$$

$$=\sum_{n=0}^{\infty}\{(1-x^2)\,P_n''-2xP_n'+n(n+1)\,P_n\}\,z^n=0.$$

[As the expression within brackets is zero for every value of n because of $P_n(x)$ satisfying Legendre's equation whatever integral value n may have.]

This proves what was required.

Ex. 13. *If (u, ϕ, z) and (r, θ, ϕ) be the cylindrical and polar co-ordinates of the same point and if $\mu=\cos\theta$, show that*

$$P_n(\mu)=(-1)^n\frac{r^{n+1}}{n!}\left(\frac{\partial}{\partial z}\right)^n\left(\frac{1}{r}\right).$$

(London B. Sc. 1926)

We know $\quad r=(x^2+y^2+z^2)^{1/2}=(u^2+z^2)^{1/2}.$

$$\therefore \quad \frac{1}{r}=(u^2+z^2)^{-1/2}=f(u, z),\ \text{say},$$

so that

$$f(u, z-k)=\{u^2+(z-k)^2\}^{-1/2}$$

$$=f(u, z)-k\frac{\partial}{\partial z}f(u, z)+\frac{k^2}{2!}\left(\frac{\partial}{\partial z}\right)^2 f(u, z)-\ldots$$

$$\ldots+(-1)^n\frac{k^n}{n!}\left(\frac{\partial}{\partial z}\right)^n f(u, z)+\ldots$$

$$\ldots+\ldots\text{[by Taylor's theorem].} \quad \ldots\text{(A)}$$

But, $f(u, z-k)=\{u^2+(z-k)^2\}^{-1/2}$

[by definition of $f(u, z)$]

$$=\{u^2+z^2-2zk+k^2\}^{-1/2}$$

$$=(r^2-2r\cos\theta . k+k^2)^{-1/2}$$

$$=(r^2)^{-1/2}\left(1-2\cos\theta\frac{k}{r}+\frac{k^2}{r^2}\right)^{-1/2}.$$

$$=\frac{1}{r}\,\Sigma\left(\frac{k}{r}\right)^n P_n(\cos\theta). \quad [\text{Art. } 1{\cdot}2] \quad \ldots(B)$$

Equating coefficients of k^n from (B) and (A), we have

$$\frac{1}{r}\cdot\frac{1}{r^n}P_n(\cos\theta)=(-1)^n\frac{1}{n\,!}\left(\frac{\partial}{\partial z}\right)^n f(u,z)$$

or
$$P_n(\cos\theta)=(-1)^n\frac{r^{n+1}}{n\,!}\left(\frac{\partial}{\partial z}\right)^n\frac{1}{r}$$

$$\left[\text{s ince } \frac{1}{r}=f(u,z)\right].$$

Ex. 14. *Prove that*

$$P_n(-\tfrac{1}{2})=P_0(-\tfrac{1}{2})\,P_{2n}(\tfrac{1}{2})+P_1(-\tfrac{1}{2})\,P_{2n-1}(\tfrac{1}{2})+\ldots$$
$$+P_{2n}(-\tfrac{1}{2})\,P_0(\tfrac{1}{2})$$

We have $(1+z^2+z^4)^{-1/2}=(1+z+z^2)^{-1/2}\,(1-z+z^2)^{-1/2}$

or
$$\{1-2(-\tfrac{1}{2})z^2+z^4\}^{-1/2}=\{1-2(-\tfrac{1}{2})z+z^2\}^{-1/2}$$
$$\times\{1-2(\tfrac{1}{2})z+z^2\}^{-1/2}$$

or
$$\sum_{n=0}^{\infty}P_n(-\tfrac{1}{2})\,(z^2)^n=\sum_{n=0}^{\infty}P_n(-\tfrac{1}{2})\,z^n\times\sum_{n=0}^{\infty}P_n(\tfrac{1}{2})\,z^n$$

or
$$P_0(-\tfrac{1}{2})+P_1(-\tfrac{1}{2})\,z^2+\ldots+P_n(-\tfrac{1}{2})\,z^{2n}+\ldots$$
$$=\{P_0(-\tfrac{1}{2})+P_1(-\tfrac{1}{2})\,z+P_2(-\tfrac{1}{2})\,z^2+\ldots$$
$$\ldots+P_n(-\tfrac{1}{2})\,z^n+\ldots+P_{2n}(-\tfrac{1}{2})\,z^{2n}+\ldots\}$$
$$\times\{P_0(\tfrac{1}{2})+P_1(\tfrac{1}{2})\,z+P_2(\tfrac{1}{2})\,z^2+\ldots$$
$$\ldots+P_n(\tfrac{1}{2})\,z^n+\ldots+P_{2n}(\tfrac{1}{2})\,z^{2n}+\ldots\}$$

Equating coefficients of z^{2n} on both sides, we have

$$P_n(-\tfrac{1}{2})=P_0(-\tfrac{1}{2})\,P_{2n}(\tfrac{1}{2})+P_1(-\tfrac{1}{2})\,P_{2n-1}(\tfrac{1}{2})$$
$$+P_2(-\tfrac{1}{2})\,P_{2n-2}(\tfrac{1}{2})+\ldots+\ldots$$
$$+P_{2n}(-\tfrac{1}{2})\,P_0(\tfrac{1}{2})$$

Exercise 1 (7)

1. Show that $\int_{-1}^{1}\mu P_n{}^2=0$

[Use result of Art. 1·6 with n for $n-1$.]

2. Show that $\int_{-1}^{1}\mu P_nP_{n-1}=\frac{2n}{4n^2-1}$

[Use result of Art. 1·6 as it is.]

3. Show that

$$1+\frac{P_1(\cos\theta)}{2}+\frac{P_2(\cos\theta)}{3}+\frac{P_2(\cos\theta)}{4}+\ldots=\log\frac{1+\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

[Integrate $\sum\limits_{n=0}^{\infty} h^n P_n(\cos\theta)=(1-2\cos\theta\, h+h^2)^{-1/2}$ with respect to h between the limits 0 to 1.]

4. Show that $\displaystyle\int_{-1}^{1}(x^2-1)\,P_n' P_{n+1}\,dx=\frac{2n(n+1)}{(2n+1)(2n+3)}$.

5. Prove that $\displaystyle\int_{-1}^{1}(1-\mu^2)P'_m(\mu)\,P_n'(\mu)=0$, where m and n are distinct positive integers. **(Agra Final 1959)**

6. Show that $\displaystyle\int_{-1}^{1}\mu P_n P_n'\,d\mu=\frac{2n}{2n+1}$.

[Take the help of recurrence formulas III and VI Arts. 1·62, 1·63.]

7. Show that $\displaystyle\int_{-1}^{1}x^2 P_n^2\,dx=\frac{2}{(2n+1)^2}\left\{\frac{(n+1)^2}{2n+3}+\frac{n^2}{2n-1}\right\}$

$$=\frac{1}{8(2n-1)}+\frac{3}{4(2n+1)}+\frac{1}{8(2n+3)}.$$

8. Show that $\displaystyle\int_{0}^{\pi}P_{2m}(\cos\theta)\,d\theta=\left(\frac{1.3.5\ldots 2m-1}{2.4.6\ldots 2m}\right)^2\pi$

$$=\left\{\frac{\Gamma\left(\frac{2m+1}{2}\right)}{\Gamma(m+1)}\right\}^2$$

9. Show that $P'_{2n+1}(\mu)$ can be expressed in the following two forms :—

(i) $(4n+1)P_{2n}+(4n-3)P_{2n-2}+\ldots+5P_2+1$,

(ii) $(2n+1)P_{2n}+2n\mu P_{2n-1}+(2n-1)\mu^2 P_{2n-2}+\ldots$
$\ldots+2\mu^{2n-1}P_1+\mu^{2n}$.

[In (ii) make repeated use of IV of Art. 1·62.]

10. $\int_{-1}^{1} P_n'(x) P_{n-1}(x)\, dx = 2.$

11. $\int_{-1}^{1} P_u'(x) P_{n-s}(x)\, dx = 2$ where s is an odd positive integer.

[In Q.'s 9—11, use Art. 1·63, (VI).]

12. $\int_{-1}^{1} P_n'(x) P_{n-s}(x)\, dx = 0$, if s is an even integer.

CHAPTER II

PART I

Integration of Bessel's equation in series for n=0. Definition of $J_0(x)$. $J_0(x)$ expressed as an integral

2·1. Integration of $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+y=0$ in series.

This equation is known as **Bessel's equation for n=0,** the general form of the Bessel's equation being

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0.$$

Let $y=\sum\limits_{r=0}^{\infty} a^r x^{k+r}$; then $\dfrac{dy}{dx}=\sum\limits_{r=0}^{\infty} a_r (k+r)x^{k+r-1}$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}.$$

Substituting in the given differential equation namely in $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+y=0$, we have

$$\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}+\sum_{r=0}^{\infty} e_r(k+r)x^{k+r-2}$$
$$+\sum_{r=0}^{\infty} a_r x^{k+r}\equiv 0$$

or, $$\sum_{r=0}^{\infty} a_r(k+r)(\overline{k+r-1}+1)x^{k+r-2}+\sum_{r=0}^{\infty} a_r x^{k+r}\equiv 0$$

or, $$\sum_{r=0}^{\infty} a_r(k+r)^2 x^{k+r-2}+\sum_{r=0}^{\infty} a_r x^{k+r}\equiv 0. \qquad \ldots(E)$$

Equating to zero the coefficient of x^{k-2} which is the lowest power of x, we have

$$a_0k^2=0, \text{ which gives } k=0, \text{ since } a_0\neq 0. \qquad \ldots(1)$$

Equating to zero the coefficient of x^{k-1}, which is the next higher power of x, we have

$$a_1\ (1+k)^2=0,$$

which gives $a_1=0$, since by virtue of (1), $(1+k)\neq 0$. ...(2)

Equating to zero the coefficient of x^{k+r}, which occurs in the general term in (E), we get

$$a_{r+2}\ (k+r+2)^2+a_r=0. \qquad \ldots(3)$$

This gives $a_{r+2}=-\dfrac{1}{(r+2)^2}\,a_r$ ...(4)

as $k=0$ from (1).

Now $a_1=0$, from (2). Hence from (4),

$$a_1=a_3=a_5=a_{2r+1}=\ldots=\ldots=0 \qquad \ldots(5)$$

Giving to r values 0, 2, 4, 6, ... we get from (4) all the a's with even suffixes in terms of a_0 which is an arbitrary constant. Thus $a_2=-\dfrac{a_0}{2^2}$, $a_4=\dfrac{a_0}{4^2.2^2}$, $a_6=-\dfrac{a_0}{6^2.4^2.2^2}$, ...

We have $y=\sum\limits_{r=0}^{\infty} a_r\, x^{k+r}$

$$=a_0\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots\right) \qquad \ldots(F)$$

because $k=0$, from (1); $a_1=a_3=a_5=\ldots=0$ from (5).

2·11. Definition of J_0 [x].

$J_0(x)$ is that solution of the differential equation $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+y=0$, which is equal to 1 for $x=0$. $J_0(x)$ is called Bassel Function of the zeroeth order.

2·12. Series for J_0 (x).

The series in (F) of article 2·1 becomes a_0 when $x=0$. Hence from the definition of $J_0(x)$, $a_0=1$, if the series in (F) of 2·1 is going to be a series of J_0 (x). Hence substituting this value of a_0 in the series, we have

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\cdots \qquad \ldots(F)$$

We notice that in (1) of 2·1, the equation to determine k, namely $a_0k^2=0$, gives two coincident values of k each being equal to zero, we are thus in possession of only one solution of the differential equation with which we started.

2·13. To show that $y=\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi$ satisfies the differential equation $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$, **(Agra 1955)**
and that y is no other than $J_0(x)$.

We have $y=\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi$. $\qquad\ldots(1)$

Differentiating under the sign of integration,

$$\frac{dy}{dx}=\frac{1}{\pi}\int_0^\pi -\sin(x\cos\phi)\cos\phi\,d\phi, \qquad \ldots(2)$$

$$\frac{d^2y}{dx^2}=-\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\cos^2\phi\,d\phi. \qquad \ldots(3)$$

By evaluating the right hand member of (2) by the method of Integration by Parts, we have

$$\frac{dy}{dx}=\frac{1}{\pi}\left[\left\{-\sin(x\cos\phi)\sin\phi\right\}_0^\pi-\int_0^\pi \cos(x\cos\phi).x\sin^2\phi\,d\phi\right]$$

$$=-\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi).x\sin^2\phi\,d\phi \qquad \ldots(4)$$

$$=-\frac{x}{\pi}\int_0^\pi \cos(x\cos\phi)(1-\cos^2\phi)\,d\phi$$

$$=-\frac{x}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi+\frac{x}{\pi}\int_0^\pi \cos(x\cos\phi)\cos^2\phi\,d\phi$$

$$=-xy-x\frac{d^2y}{dx^2} \text{ [from (1) and (3)];}$$

$$\therefore \quad \frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0 \quad \text{[dividing throughout by } x\text{].}$$

Thus $y=\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi$ satisfies the given differential equation, which is Bessel's equation of the zeroeth order and since this integral for y has 1 for its value when $x=0$, it is no other than $J_0(x)$, as $J_0(x)$ being the solution of Bessel's equation of the zeroeth order is equal to 1 for $x=0$.

Exercises 2 (1)

1. In result (1) of article 2·1, can we say $a_0=0$, $k\neq 0$? Give reasons for the answer.

   [**Ans.** a_0 being the coefficient of the very first term can not be zero because with whatever term you begin to write the series that term is the first term and if the coefficient of this term is zero, you can not write the first term, that is, you can not begin to write the series in other words you can not write the series at all. Hence $a_0\neq 0$, and so, $k=0$.

2. Put $\cos(x\cos\phi)$ as a series in powers of $x\cos\phi$, and then integrate with respect to ϕ between the limits 0 to π for ϕ, x being independent of ϕ. Do the same with $\cos(x\sin\phi)$, and thus show that

   $$\frac{1}{\pi}\int_0^\pi \cos(x\cos\phi)\,d\phi=\frac{1}{\pi}\int_0^\pi \cos(x\sin\phi)\,d\phi=J_0(x).$$

CHAPTER II

PART 2

Integration of Bessel's general equation in series, Bessel's function.

2·2. Let us integrate Bessel's general differential equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0. \qquad \ldots(1)$$

Put $y=\sum_{r=0}^{\infty} a_r x^{k+r}$; then

$$\frac{dy}{dx}=\sum_{r=0}^{\infty} a_r (k+r)\, x^{k+r-1},$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty} a_r (k+r)\,(k+r-1)\, x^{k+r-2}.$$

Substituting these in (1), we have

$$\sum_{r=0}^{\infty} a_r (k+r)\,(k+r-1)\, x^{k+r-2}+\sum_{r=0}^{\infty} a_r (k+r)\, x^{k+r-2}$$

$$+\sum_{r=0}^{\infty} a_r x^{k+r}-\sum_{r=0}^{\infty} a_r n^2 x^{k+r-2}\equiv 0$$

or $$\sum_{r=0}^{\infty} a_r \{(k+r)^2-n^2\}\, x^{k+r-2}+\sum_{r=0}^{\infty} a_r x^{k+r}\equiv 0. \qquad \ldots(S)$$

Equating to zero the coefficient of x^{k-2} which is the lowest power of x in (S), we have

$a_0\,(k^2-n^2)=0$ which gives $k=\pm n$, since $a_0\neq 0$. $\quad \ldots(1)$

Equating to zero the coefficient of x^{k-1} which is the next higher power of x in (S), we have

$$a_1\{(1+k)^2-n^2\}=0.$$

As $(1+k)^2-n^2\neq 0$ by virtue of (1),

so $$a_1=0. \qquad \ldots(2)$$

Equating to zero the coefficient of x^{k+r} in (S), we get

$$a_{r+2}\{(k+r+2)^2-n^2\}+a_r=0 \qquad \ldots(3)$$

or $$a_{r+2}=-\frac{a_r}{(n+r+2)^2-n^2} \text{ for } k=n \qquad \ldots(4)$$

$$a_{r+2}=-\frac{a_r}{(-n+r+2)^2-n^2} \text{ for } k=-n. \qquad \ldots(5)$$

From (4), $$a_{r+2}=-\frac{a_r}{(r+2)\,(2n+r+2)}. \qquad \ldots(6)$$

From (2) and (6), $a_1=a_3=a_5=\ldots=0.$

Giving to r values 0, 2, 4, 6,...we get the corresponding a's which on substitution give us our solution as the series

$$y=a_0x^n\left[1-\frac{x^2}{2.2\,(n+1)}+\frac{x^4}{2.4.2^2\,(n+1)\,(n+2)}-\ldots\right.$$
$$\left.+(-1)^r\frac{x^{2r}}{2^r.r\,!.2^r.(n+1)\,(n+2)\ldots(n+r)}+\ldots\right], \ldots(7)$$

where a_0 is an arbitrary constant.

The other solution is obtained by replacing n by $-n$ in (7).

Definition of J_n (x). **(Agra 1952, 54, 55)**

The solution as given by (7) is called $J_n(x)$, when

$$a_0=\frac{1}{2^n\Gamma(n+1)},$$

where, as we know, $\Gamma(n+1)=n\,!$, if n is a positive integer.

In fact $J_n(x)$ is that solution of the general Bessel's equation, which behaves as $\dfrac{x^n}{2^n\Gamma(n+1)}$ for values of x.

so that, $$J_n(x)=\frac{x^n}{2^n\,\Gamma(n+1)}\left[1-\frac{x^2}{2.2(n+1)}+\frac{x^4}{2.4.2^2(n+1)(n+2)}+\ldots\right].$$

or $$\mathbf{J}_n(\mathbf{x})=\frac{x^n}{2^n.\Gamma(n+1)}\left[\sum_{r=0}^{\infty}(-1)^r.\left(\frac{x}{2}\right)^{2r}\times\frac{1}{r!.(n+1)(n+2)\ldots(n+r)}\right]$$

$$=\sum_{r=0}^{\infty}(-1)^r\left(\frac{\mathbf{x}}{2}\right)^{n+2r}\frac{1}{\mathbf{r}!.\Gamma(\mathbf{n}+\mathbf{r}+1)}. \qquad \ldots(8)$$

and $$\mathbf{J}_{-n}(\mathbf{x})=\sum_{r=0}^{\infty}(-1)^r\left(\frac{\mathbf{x}}{2}\right)^{-n+2r}\frac{1}{\mathbf{r}!.\Gamma(-\mathbf{n}+\mathbf{r}+1)} \qquad \ldots(9)$$

$\mathbf{J}_n(\mathbf{x})$ as defined above is called **Bessel's function of the First kind of order n.**

The complete primitive of Bessel's equation is
$$AJ_n(x)+BJ_{-n}(x),$$
if n is not an integer.

2·22. To prove that $\mathbf{J}_{-n}(\mathbf{x})=(-1)^n\,\mathbf{J}_n(\mathbf{x})$, n being integral and positive.

We have, $$J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{r!\,\Gamma(-n+r+1)}.$$

Here, in the denominator of all the terms which precede the $(n+1)$th, we have for the argument of Γ either zero or a negative integer, thus making $\Gamma(0)=\Gamma(-N)=\infty$, where $-N$ is a negative integer, hence all the terms which precede the $(n+1)$th vanish as the denominator in each case becomes infinite. Hence

$$J_{-n}(x)=\sum_{r=n}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{-n+2r}\frac{1}{r!.\Gamma(-n+r+1)}$$

$$=\sum_{s=0}^{\infty}(-1)^{s+n}\left(\frac{x}{2}\right)^{-n+2(s+n)}\frac{1}{(n+s)!.\Gamma(-n+s+n+1)}$$

$$=\sum_{s=0}^{\infty}(-1)^{s+n}\left(\frac{x}{2}\right)^{2s+n}\frac{1}{(n+s)!.\Gamma(s+1)}$$

$$=(-1)^n\sum_{s=0}^{\infty}(-1)^{s}\left(\frac{x}{2}\right)^{n+2s}\frac{1}{s!.\Gamma(n+s+1)}$$

$$=(-1)^n J_n(x).$$

Exercises 2 (2)

1. Is the result $J_{-n}(x)=(-1)^n J_n(x)$ universelly true ?

   Ans. No. it is true only for integral values of n.

2. Do you think that you have completely solved Bessel's equation for the case when n is an integer ?

   Ans. No. The two solutoons J_n and J_{-n} being not independent due to the relation $J_{-n}(x)=(-1)^n J_n(x)$, a further investigation for the second solution is needed.

3. Prove that $J_{\frac{1}{2}}(x)=\sqrt{\left(\frac{2}{\pi x}\right)}\sin x.$ **(Agra, 1963)**

   $\Bigg[$We know $J_n(x)=\dfrac{x^n}{2^n\Gamma(n+1)}\left\{1-\dfrac{x^2}{2.2(n+1)}+\dfrac{x^4}{2.4.2^2(n+1)(n+2)}-\dots\right\}$

   so that $$J_{\frac{1}{2}}(x)=\left(\frac{x}{2}\right)^{\frac{1}{2}}\frac{1}{\Gamma(\frac{3}{2})}\left\{1-\frac{x^2}{2.2.\frac{3}{2}}+\frac{x^4}{2.4.2^2.\frac{3}{2}.\frac{5}{2}}-\dots\right\}$$

   $$=\left(\frac{x}{2}\right)^{\frac{1}{2}}\frac{1}{\frac{1}{2}\Gamma(\frac{1}{2})}\left\{1-\frac{x^2}{2.3}+\frac{x^4}{2.4.3.5}-\dots\right\}$$

   $$=\frac{\sqrt{x}\sqrt{2}}{\Gamma(\frac{1}{2})}.\frac{1}{x}\left\{x-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots\right\}$$

   $$=\sqrt{\left(\frac{2}{\pi x}\right)}\sin x\ [\text{as } \Gamma(\tfrac{1}{2})=\sqrt{\pi}]\Bigg].$$

4. Prove that $J_{-\frac{1}{2}}(x)=\dfrac{2}{\sqrt{(\pi x)}}\cos x.$

CHAPTER II

PART 3

Recurrence formulae for $J_n(x)$

2·3. n being a positive integer, we have

$$J_n(x)=\sum_{r=0}^{\infty} \frac{(-1)^r}{r!.(n+r)!}\left(\frac{x}{2}\right)^{n+2r} \qquad \text{[By (8) of Art. 2·2]}$$

$$\therefore \quad xJ_n'(x)=\sum_{r=0}^{\infty} \frac{(-1)^r (n+2r)}{r!.(n+r)!}\cdot\left(\frac{x}{2}\right)^{n+2r} \qquad \ldots(1)$$

$$=\sum_{r=0}^{\infty} (-1)^r \frac{n}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$+\sum_{r=0}^{\infty} \frac{(-1)^r.2r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

$$=nJ_n(x)+x\sum_{r=1}^{\infty} \frac{(-1)^r}{(r-1)!.(n+r)!}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=nJ_n(x)-x\sum_{s=0}^{\infty} \frac{(-1)^s}{s!.(n+s+1)!}\left(\frac{x}{2}\right)^{n+2s+1}$$

[Putting $r-1=s$]

$$=nJ_n(x)-xJ_{n+1}(x). \qquad \ldots(A)$$

Again, (1) can be written as,

$$xJ_n'(x)=\sum_{r=0}^{\infty} \frac{(-1)^r[(-n)+2(n+r)]}{r!.(n+r)!}\left(\frac{x}{2}\right)^{n+2r}$$

or $$xJ_n'(x)=-nJ_n(x)+x\sum_{r=0}^{\infty} \frac{(-1)^r}{r!.(n+r-1)!}\left(\frac{x}{2}\right)^{n+2r-1}$$

$$=-nJ_n(x)+xJ_{n-1}(x). \qquad \ldots(B)$$

Thus, we have from (A) and (B),

$$xJ_n'(x)=nJ_n-xJ_{n+1}, \qquad \ldots\text{(I)}$$

(Agra 1954)

$$xJ_n'(x)=-nJ_n+xJ_{n-1}. \qquad \ldots\text{(II)}$$

(Agra 1954)

From the above we get by subtracting and adding

$$2nJ_n=x(J_{n-1}+J_{n+1}), \qquad \ldots\text{(III)}$$

(Agra 1952, 56)

$$2J_n'(x)=J_{n-1}-J_{n+1}. \qquad \ldots\text{(IV)}$$

(Agra 1963, 58)

2·31. To prove

$$\frac{\mathbf{d}}{\mathbf{dx}}\{\mathbf{x}^n\mathbf{J}_n(\mathbf{x})\}=\mathbf{x}^n\mathbf{J}_{n-1}(\mathbf{x}). \qquad \textbf{(Agra 1957)}$$

Now $\dfrac{d}{dx}\{x^nJ_n(x)\}=nx^{n-1}J_n(x)+x^nJ_n'(x)$

$$=x^{n-1}\{nJ_n(x)+xJ_n'(x)\}$$

$$=x^{n-1}\{nJ_n(x)-nJ_n(x)+xJ_{n-1}(x)\}$$

[By (II) of Art. 2·3]

$$=x^nJ_{n-1}(x). \qquad \ldots\text{(V)}$$

To prove

$$\frac{\mathbf{d}}{\mathbf{dx}}\{\mathbf{x}^{-n}\mathbf{J}_n(\mathbf{x})\}=-\mathbf{x}^{-n}\mathbf{J}_{n+1}(\mathbf{x}).$$

We have

$$\frac{d}{dx}\{x^{-n}J_n(x)\}=-nx^{-n-1}J_n(x)+x^{-n}J_n'(x)$$

$$=x^{-n-1}\{-nJ_n(x)+xJ_n'(x)\}$$

$$=x^{-n-1}\{-nJ_n(x)+nJ_n(x)-xJ_{n+1}\}$$

[By (I) of Art. 2·3]

$$=-x^{-n}J_{n+1}(x). \qquad \ldots\text{(VI)}$$

Exercises 2 (3)

1. Show that

(i) $J_0'(x)=-J_1(x)$, and (ii) $2J_0''(x)=J_2(x)-J_0(x)$.

[To get (ii), differentiate (i) and apply (IV) Art. 2·3].

2. Prove that

$$J_{n+3}+J_{n+5}=\frac{2}{x}(n+4)\,J_{n+4}. \qquad \textbf{(Agra 53)}$$

[Put $(n+4)$ for n in IV Art. 2·3].

3. Show that

$$\frac{d}{dx}\left(J^2{}_n+J^2{}_{n+1}\right)=2\left(\frac{n}{x}J^2{}_n-\frac{n+1}{x}J^2{}_{n+1}\right)$$

(Agra final 1947)

[Use recurrence formulae (I) and (II) of Art. 2·3 in evaluating the left hand side].

CHAPTER II

PART 4

Generating function for $J_n(x)$.

Certain Series involving $J_n(x)$.

2·4. Generating function for $J_n(x)$.

To show that $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{-\infty}^{\infty} J_n(x)\,t^n.$

(Agra 1963, 1952)

We know.

$$e^{\frac{x}{2}t}=\sum_{r=0}^{\infty}\frac{1}{r\,!}\left(\frac{x}{2}\right)^r t^r, \qquad \ldots(A)$$

$$e^{-\frac{x}{2}\cdot\frac{1}{t}}=\sum_{s=0}^{\infty}\frac{1}{s\,!}\left(-\frac{x}{2}\right)^s\frac{1}{t^s}. \qquad \ldots(B)$$

Hence by multiplication,

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{x}{2}\right)^r t^r\times\sum_{s=0}^{\infty}\frac{1}{s\,!}\left(-\frac{x}{2}\right)^s\frac{1}{t^s}. \qquad \ldots(C)$$

The sum of terms involving t^n in the product of the two series on right side of (C) for any given value of n, consists of all terms formed by multiplying each term $\frac{1}{s\,!}\left(-\frac{x}{2}\right)^s\frac{1}{t^s}$ of the series (B), with the term $\frac{1}{(n+s)\,!}\cdot\left(\frac{x}{2}\right)^{n+s}t^{n+s}$ of the series (A), where s takes values ranging from 0 to ∞.

Hence the sum of all terms involving t^n in the product of the two series

$$=\sum_{s=0}^{\infty}\left\{\frac{1}{s\,!}\left(-\frac{x}{2}\right)^s\frac{1}{t^s}\right\}\left\{\frac{1}{(n+s)\,!}\left(\frac{x}{2}\right)^{n+s}t^{n+s}\right\},$$

so that the coefficient of t^n in this product of the two series in (C)

$$=\sum_{s=0}^{\infty}\frac{(-1)^s}{s\,!\,(n+s)\,!}\cdot\left(\frac{x}{2}\right)^{n+2s}=J_n(x). \qquad \ldots(F_1)$$

Similarly, the coefficient of $\frac{1}{t^n}$ is obtained from terms obtained by multiplying each term $\frac{x^r}{2^r\,r\,!}\,t^r$ of the series (A) with the term $\frac{(-x)^{n+r}}{2^{n+r}\,(n+r)\,!}\cdot\frac{1}{t^{n+r}}$ of the series (B), where r ranges from 0 to ∞.

Hence the coefficient of $\frac{1}{t^n}$ or of t^{-n} in the product of rhe two series in (C) is

$$\sum_{r=0}^{\infty}(-1)^{n+r}\,\frac{1}{r\,!\,(n+r)\,!}\cdot\left(\frac{x}{2}\right)^{n+2r}$$

$$=(-1)^n\sum_{r=0}^{\infty}(-1)^n\,\frac{1}{r\,!\,(n+r)\,!}\cdot\left(\frac{x}{2}\right)^{n+2r}$$

$$=(-1)^n J_n(x)=J_{-n}(x) \quad \text{[Art. 2·22].} \qquad \ldots(F_2)$$

In view of (C), (F_1), and (F_2) we have

$$\mathbf{e}^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{-\infty}^{\infty}\mathbf{J}_n(\mathbf{x}).\mathbf{t}^n. \qquad \ldots(F)$$

This is exactly what we had to prove.

Because of their appearance as coefficients in the above expansion, (F) the functions, J_n of integral order are known as **Bassel's Coefficients.**

2·41. Series for $\begin{matrix}\cos\\ \sin\end{matrix}(x \sin \phi)$, $\begin{matrix}\cos\\ \sin\end{matrix}(x \cos \phi)$.

By putting $t=e^{ix}$ in (F) of Art. 2·4, we have

$$e^{i\varphi \sin \varphi}=\sum_{n=-\infty}^{\infty} e^{in\varphi} J_n(x)$$

or $\cos(x \sin \phi)+i \sin(x \sin \phi)$

$$=J_0(x)+[J_1(x)\, e^{i\varphi}+J_{-1}(x)\, e^{-i\varphi}]$$
$$+[J_2(x)\, e^{2i\varphi}+{}_{-}J_2(x)^{-2i\varphi}$$
$$+[J_3(x)\, e^{3i\varphi}+{}_{-}J_3(x)\, e^{-3i\varphi}]+\ldots$$
$$=J_0(x)+J_1(x)\,[e^{i\varphi}-e^{-i\varphi}]$$
$$+J_2(x)\,[e^{2i\varphi}+e^{-2i\varphi}]+J_3(x)\,[e^{3i\varphi}-e^{-3i\varphi}]$$
$$+J_4(x)\,[e^{4i\varphi}+e^{-4i\varphi}]+\ldots\ldots$$

[because $J_{-n}(x)=(-1)^n J_n(x)$]

$$=(J_0(x)+2i \sin \phi\, J_1(x)+2 \cos 2\phi\, J_2(x)$$
$$+2i \sin 3\phi\, J_3(x)+2 \cos 4\phi\, J_4(x)+\ldots \quad \ldots(F)$$

Equating real and imaginary parts in (F), we have

$$\cos(x \sin \phi)=J_0(x)+2 \cos 2\phi\, J_2(x)+2 \cos 4\phi\, J_4(x)+\ldots \quad \ldots(1)$$

$$\sin(x \sin \phi)=2J_1(x) \sin \phi+2J_3(x) \sin 3\phi + 2J_5(x) \sin 5\phi+\ldots \quad \ldots(2)$$

By replacing ϕ by $\frac{\pi}{2}-\phi$ in (1) and (2), we get

$$\cos(x \cos \phi)=J_0(x)-2 \cos 2\phi\, J_2(x)+2 \cos 4\phi\, J_4(x) -2 \cos 6\phi\, J_6(x)+\ldots \quad \ldots(3)$$

$$\sin(x \cos \phi)=2 \cos \phi\, J_1(x)-2 \cos 3\phi\, J_3(x) +2 \cos 5\phi\, J_5(x)-2 \cos 7\phi\, J_7(x)+\ldots \quad \ldots(4)$$

Exercises 2 (4)

1. Show that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \ldots$$
$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \ldots$$

(Agra 1936)

[**Hint.** Put $\phi=0$ in (3) and (4) of Art 2·41].

2. $x \sin x = 2\,[2^2 J_2(x) - 4^2 J_4(x) + 6^2 J_6(x) - \ldots]$,
$x \cos x = 2\,[1^2 J_1(x) - 3^2 J_3(x) + 5^2 J_5(x) - \ldots]$.

[**Hint.** Differentiate (3) and (4) of Art. 2·41 twice with respect to ϕ, and then put $\phi=0$].

3. Show that $\int_0^x x^{-n} J_{n+1}(x)\,dx = \dfrac{1}{2^n\,\Gamma(n+1)} - x^n J_n$, where $n > -1$. [Use (VI) of Art. 2·31]

4. Show that $\int_0^x x^n J_{n-1}(x)\,dx = x^n J_n(x)$.

[Use (V) of Art. 2·31]

CHAPTER II

PART 5

Integrals for $J_0(x)$ and $J_n(x)$

2·5. From (1) of Art. 2·41, we get by integrating with respect to ϕ

$$\int_0^{\pi} \cos(x \sin\phi)\, d\phi = \pi J_0(x),$$

as other terms on the right hand side vanish between the limits of integration.

$$\therefore \quad J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\phi)\, d\phi. \qquad \ldots(A)$$

Again multiplying (1) of Art. 2·41 by $\cos n\phi$ and integrating, we get

$$\int_0^{\pi} \text{css}\,(x \sin\phi) \cos n\phi \, d\phi = 0 \quad \text{or} \quad \pi J_n(x), \quad \ldots(1)$$

according as n is odd or even.

Multiplying (2) of Art. 2·41 by $\sin n\phi$ and intograting, we have

$$\int_0^{\pi} \sin(x \sin\phi) \sin n\phi \, d\phi = 0 \quad \text{or} \quad \pi J_n(x), \quad \ldots(2)$$

according as n is even or odd.

Hence on adding (1) and (2), we get

$$\int_0^{\pi} \{\cos(x \sin\phi) \cos n\phi + \sin(x \sin\phi) \sin n\phi\}\, d\phi$$

$$= \pi J_n(x), \text{ whether } n \text{ is odd or even}$$

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - x \sin\phi)\, d\phi. \qquad \ldots(F)$$

whether n is odd or even. **(Agra 1962, 58. 56, 52)**

2·51. To prove

$$J_n(x)=\frac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_0^\pi \cos(x\sin\phi)\cos^{2n}\phi\,d\phi. \quad \ldots(1)$$

Expanding $\cos(x\sin\phi)$ in powers of $x\sin\phi$, the genera l term on the right hand side is

$$\frac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\cdot\left(\frac{x}{2}\right)^n(-1)^r\int_0^\pi \frac{x^{2r}}{(2r)!}\sin^{2r}\phi\cos^{2n}\phi\,d\phi.$$

Now $\displaystyle\int_0^\pi \sin^{2r}\phi\cos^{2n}\phi\,d\phi=2\int_0^{\pi/2}\sin^{2r}\phi\cos^{2n}\phi\,d\phi$

$$=\int_0^1 t^{(2r-1)/2}(1-t)^{(2n-1)/2}\,dt$$

where $t=\sin^2\phi$;

$$=\frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(1+r+n)}.$$

Hence the general term on the right hand side of (1) is

$$=\frac{1}{\sqrt{\pi}}\cdot\frac{1}{\Gamma(n+\frac{1}{2})}\cdot\left(\frac{x}{2}\right)^n(-1)^r\cdot\frac{x^{2r}}{(2r)!}$$

$$\times\frac{\frac{2r-1}{2}\cdot\frac{2r-3}{2}\ldots\frac{1}{2}\,\Gamma(\frac{1}{2})\;\Gamma(n+\frac{1}{2})}{\Gamma(1+r+n)}$$

$$=(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r\,!}\,\frac{1}{\Gamma(n+r+1)}=\text{general term of } J_n.$$

Thus, we have

$$J_n(x)=\frac{1}{\sqrt{\pi}\Gamma(n+\frac{1}{2})}\left(\frac{x}{2}\right)^n\int_0^\pi \cos(x\sin\phi)\cos^{2n}\phi\,d\phi.$$

2·52. Investigation of the solution $J_n(x)$ of the Bessel's differential equation in the form

$$J_n(x)=(-2)^n x^n\frac{d^n}{d(x^2)^n}J_0(x). \quad \textbf{(Agra 1958)}$$

We take the differential equation for $J_0(x)$ and change the independent variable in a certain manner and differentiate n times and then we take the differential equation for $J_n(x)$ and change the independent and the dependent variables in certain ways and then by comparison we get the result thus.

Consider the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0 \qquad \ldots(1)$$

$J_0(x)$ is its solution. Changing the independent variable from x to X, by the relation $x^2=X$, we have

$$\frac{dy}{dx}=\frac{dy}{dX}\cdot\frac{dX}{dx}=2x\frac{dy}{dX}$$

$$\frac{d^2y}{dx^2}=2\frac{dy}{X}+2x\frac{d^2y}{dX^2}\cdot\frac{dX}{dx}$$

$$=2\frac{dy}{dX}+4x^2\frac{d^2y}{dX^2}.$$

Substituting in (1), we have

$$4X\frac{d^2y}{dX^2}+4\frac{dy}{dX}+y=0. \qquad \ldots(2)$$

Differentiating (2), n times with respect to X, we get

$$4X\frac{d^{n+2}y}{dX^{n+2}}+4(n+1)\frac{d^{n+1}y}{dX^{n+1}}+\frac{d^ny}{dX^n}=0.$$

Putting $Y=\dfrac{d^ny}{dX^n}=\left(\dfrac{d}{dX}\right)^n J_0(x)$, $\qquad \ldots(S)$

$$4X\frac{d^3Y}{dX^2}+4(n+1)\frac{dY}{dX}+Y=0. \qquad \ldots(3)$$

Now take the equation for $J_n(x)$ namely

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left(1-\frac{n^2}{x^2}\right)y=0. \qquad \ldots(4)$$

In (4) put
$$y=x^n Z. \qquad \ldots(T)$$
then
$$\frac{d^2Z}{dx^2}+\frac{(2n+1)}{x}\frac{dZ}{dx}+Z=0. \qquad \ldots(5)$$

In (5) put $x^2=X$; it then becomes
$$4X\frac{d^2Z}{dX^2}+4(n+1)\frac{dZ}{dX}+Z=0. \qquad \ldots(6)$$

Comparing (3) and (6), we see that $Z=Y$ ...(7)

y=solution of $(4)=x^nZ=x^nY$, [From (T) and (7)]

$=x^n.\left(\frac{d}{dX}\right)^n J_0(x)$ [From (S)]

or
$$y=x^n\frac{d^n}{d(x^2)^n}J_0(x)$$
or
$$J_n(x)=C.x^n\frac{d^n}{d(x^2)^n}J_0(x) \qquad \ldots(8)$$

where C is a constant to be determined

or $J_n(x)=C.x^n\frac{d^n}{d(x^2)^n}\left(1-\frac{x^2}{2^2}+\frac{x^4}{2^2.4^2}-\frac{x^6}{2^2.4^2.6^2}+\ldots\right)$

$$=Cx^n\frac{d^n}{dt^n}\left(1-\frac{t}{2^2}+\frac{t^2}{2^2.4^2}-\frac{t^3}{2^2.4^2.6^2}+\ldots\right)$$

where $t=x^2$

$$=Cx^n.\left(\frac{d}{dt}\right)^n\sum_{r=0}^{\infty}(-1)^{n+r}\frac{t^{n+r}}{2^2.4^2\ldots(2n+2r)^2}$$

[as all powers of t with indices less than n vanish on differentiation]

$$=Cx^n.\sum_{r=0}^{\infty}(-1)^{n+r}.\frac{(n+r)(n+r-1)\ldots(r+1)\,t^r}{2^2.4^2\ldots(2n+2r)^2}$$

$$=C.x^n\sum_{r=0}^{\infty}(-1)^{n+r}.\frac{t^r}{(2^2)^{n+r}.(r\,!)^2.(r+1)(r+2)\ldots(r+n)}$$

...(9)

$=C.x^n.\frac{(-1)^n}{2^{2n}(n\,!)}$ for small values of x, only the lowest power of x in (9) having been retained.

Hence from the definition of J_n that it behaves as $\frac{x^n}{2^n\,(n)\,!}$ for small values of x (Art. 2·2), $C=(-2)^n$.

With this value of C, we have from (8)

$$J_n(x)=(-2)^n\, x^n \frac{d^n}{d\,(x^2)^n} J_0(x). \qquad \textbf{(Agra 1958)}$$

Now, putting the same value of C in (9),

$$J_n(x)=(-2)^n.x^n \sum_{r=0}^{\infty} (-1)^{n+r}.\frac{x^{2r}}{2^{2r+2n}\,(r\,!)^2\,(r+1)\,(r+2)\ldots(r+n)}$$

$$=\sum_{r=0}^{\infty} (-1)^{2n+r}.\frac{2^n x^{n+2r}}{2^{2n+2r}\,(r\,!).1.2.3\ldots(n+r)}$$

$$=\sum_{r=0}^{\infty} (-1)^r.\frac{1}{r\,!\,(n+r)\,!}.\left(\frac{x}{2}\right)^{n+3r}$$

$$=\sum_{r=1}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(n+r+1)}.\left(\frac{x}{2}\right)^{n+2r} \quad \text{[the series for } J_n]$$

Exercise 2 (5)

1. $\int_0^{\pi} \cos(x\cos\phi)\cos(2n+1)\,\phi\, d\phi=0,$

   $\int_0^{\pi} \sin(x\cos\phi)\cos 2n\phi\, d\phi=0.$

2. Show that

   (a) $\frac{(-1)^n}{\pi}\int_0^{\pi} \cos(x\cos\phi)\cos n\phi\, d\phi=J_n(x)$ or zero according as n is even or odd.

   (b) $\frac{(-1)^n}{\pi}\int_0^{\pi} \sin(x\cos\phi)\cos n\phi\, d\phi=J_n(x)$ or zero according as n is odd or even.

   [Use results (3) and (4) of Art. 2·4.]

CHAPTER II

PART 6

Integral properties of $J_n(x)$. Zeros of $J_n(x)$

2·6. To show that $\int_0^a J_n(k_r) J_n(k'r)\, r\, dr = 0$,

where k and k' are different roots of $J_n(xa)=0$.

Consider the equations

$$r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} + (k^2r^2 - n^2)\, u = 0 \qquad \ldots(1)$$

$$r^2 \frac{d^2v}{dr^2} + r\frac{dv}{dr} + (k'^2r^2 - n^2) v = 0 \qquad \ldots(2)$$

Multiplying (1) by $\frac{v}{r}$, and (2) by $\frac{u}{r}$, and subtracting,

$$\frac{d}{dr}\left\{ r\,(vu' - uv') \right\} + (k^2 - k'^2)\, ruv = 0 \qquad \ldots(3)$$

Since $u = J_n(kr)$ and $v = J_n(k'r)$ are solutions of (1) and (2), integrating (3) with respect to r between the limits from 0 to a; we have

$$\int_0^a (k^2 - k'^2)\, J_n(kr)\, J_n(k'r)\, r\, dr$$

$$= \Big[r \left\{ J_n(kr) . J_n'(k'r) . k' - J_n(k'r)\, J_n'(kr)\, k \right\} \Big]_0^a = 0$$

[since $J_n(ka)=0$, $J_n(k'a)=0$.]

Hence the theorem.

2·61. All the roots of $J_n(x)=0$ are real.

If the theorem is not true, then take two conjugate complex roots $\lambda + i\mu$ and $\lambda - i\mu$.

Put in the result of the last article,

$$ka=\lambda+i\mu,\ k'a=\lambda-i\mu,$$

Then $$4i\lambda\mu\int_0^a J_n(kr)\,J_n(k'r)\,r\,dr=0 \qquad \text{...(1)}$$

But $J_n(kr)$ and $J_n(k'r)$ are conjugate complex quantities, say, equal to $P+iQ$ and $P-iQ$ respectively.

Hence (1) becomes

$$4i\lambda\mu\int_0^a (P^2+Q^2)\,r\,dr=0.$$

Since the integral is not zero as the integrand is throughout positive, we have $\lambda\mu=0$, but λ cannot be zero as then a purely imaginary quantity $i\mu$ will have to satsify $J_n(x)=0$ which is impossible since the sum of decidedly positive quantities cannot be zero; it, therefore, follows that $\mu=0$.

Hence the theorem.

2·62. $J_n(x)=0$ **has no repeated roots except** $x=0$

(Agra 1959)

We have from (1) of Art. 2·3,

$$J'_n=\frac{n}{x}J_n-J_{n+1} \qquad \text{...(1)}$$

Suppose α is a repeated root of J_n, then

$$J_n(\alpha)=0,\ J'_n(\alpha)=0,$$

so that from (1), $$J_{n+1}(\alpha)=0,$$

and then from (III) of Art. 2·3, $J_{n-1}(\alpha)=0$

Thus for the same value α of x, $J_n(x)$, $J_{n+1}(x)$, $J_{n-1}(x)$ are all equal to zero, which is absurd as we cannot have two power series having the same sum function. Thus there cannot be any repeated root of $J_n(x)=0$ except $x=0$.

2·63. Solved Examples.

Example 1. *Show that* $\dfrac{x}{2} J_n = (n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} - \ldots$

(Agra 61, 58, 56)

From recurrence formula III Art. 2·3, we have

$$
\text{(A)}\left\{
\begin{array}{lll}
J_n + I_{n+2} &= \dfrac{2}{x}(n+1) J_{n+1} & \text{[replacing } n \text{ by } n+1 \text{ in the formula]} \\[2ex]
-(J_{n+2} + J_{n+4}) &= -\dfrac{2}{x}(n+3) J_{n+3} & \text{[putting } n+2 \text{ for } n \text{ in the above and changing the sign of both sides]} \\[2ex]
J_{n+4} + J_{n+6} &= \dfrac{2}{x}(n+5) J_{n+5} & \quad ,, \\[2ex]
-(J_{n+6} + J_{n+8}) &= -\dfrac{2}{x}(n+7) J_{n+7} & \quad ,, \\[2ex]
\ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \quad \ldots & \\
\ldots \quad \ldots & \ldots \quad \ldots \quad \ldots \quad \ldots &
\end{array}
\right.
$$

From the series for $J_n(x)$ we notice that as $\dfrac{x^n}{n!} \to 0$ as $n \to \infty$ $\left(\dfrac{x^n}{n!}\right.$ being the nth term of a convergent $\left.\text{series}\right)$, and $J_n(x) = \dfrac{x^n}{n!} \cdot \dfrac{1}{2^n} \times$ a convergent series. Taking limits, $J_\infty = 0$.

Hence on adding both sides of (A), we have

$$
J_n = \frac{2}{x}\left[(n+1) J_{n+1} - (n+3) J_{n+3} + (n+5) J_{n+5} - (n+7) J_{n+7} + \ldots\right] \quad \ldots(1)
$$

or $\quad \dfrac{xJ_n}{2}=\{(n+1)\,J_{n+1}-(n+3)\,J_{n+3}+(n+5)\,J_{n+5}-\ldots\}.$

Example 2. *Define $J_n(x)$, n being a positive integer, and prove that*

$$\frac{dJ_n}{dx}=\frac{2}{x}\left\{\frac{nJ_n}{2}-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-(n+6)\,J_{n+6}+\ldots\right\}.$$

(Agra 1955)

n being a positive integer, $J_n(x)$ is that solution of the differential equation $\dfrac{d^3y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+\left(1-\dfrac{n^2}{x^2}\right)=0$, which behaves as $\dfrac{x^n}{2^n.n\,!}$ for small values of x. In fact,

$$J_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!.\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}.$$

(Rajasthan 1959)

From recurrence formulae (II) of Art. 2·3,

$$J_n'(x)=-\frac{nJ_n}{x}+J_{n-1}$$

$$=-\frac{nJ_n}{x}+\frac{2}{x}\{nJ_n-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-(n+6)\,J_{n+6}$$

$$+\ldots\quad \text{[by result of Example 1]}$$

$$=\frac{2}{x}\left[\frac{nJ_n}{2}-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-(n+6)\,J_{n+6}+\ldots\right].$$

Ex. 3. *Prove that*

$$\{J_0(x)\}^2+2\,[\{J_1(x)\}^2+\{J_2(x)\}^2+\ldots]=1.$$

(Agra 1957)

Equalities (1) and (2) of art. 2·41 are as follows :

$$\cos(x\sin\phi)=J_0(x)+2J_2(x)\cos 2\phi+2J_4(x)\cos 4\phi+\ldots \quad \ldots(1)$$

$$\sin(x\sin\phi)=2J_1(x)\sin\phi+2J_3(x)\sin 3\phi+2J_5(x)\sin 5\phi+\ldots \quad \ldots(2)$$

Squaring and adding (1) and (2), we have

$$1=\{J_0(x)\}^2+4\,[\{J_1(x)\}^2\sin^2\phi+\{J_2(x)\}^2\cos^2 2\phi+\{J_3(x)\}^2\sin^2 3\phi+\{J_4(x)\}^2\cos^3 4\phi+\ldots]$$

+terms involving single power of $\cos n\phi$ where n is even,
+terms involving product of either two different sines or two different cosines. ...(3)

Integrating (3) with respect to ϕ between the limits 0 to π, we get

$$\int_0^\pi 1.d\phi=\int_0^\pi \{J_0(x)\}^2\,d\phi+2\int_0^\pi [\{J_1(x)\}^2.2\sin^2\phi+\{J_2(x)\}^2.2\cos^2 2\phi+\{J_3(x)\}^2.2\sin^2\phi+\{J_4(x)\}^2.2\cos^2 4\phi+\ldots]\,d\phi$$

+integrals of terms involving single powers $\cos n\phi$ or products $\cos n\phi\cos m\phi$ or products $\sin p\phi\sin q\phi$, where n and m are even positive integers and p and q are odd positive integers which, on evaluation, all vanish between limits of integration 0 to π for ϕ.

Hence,

$$\pi=\pi\,\{J_0(x)\}^2+2\pi\,[\{J_1(x)\}^2+\{J_2(x)\}^2+\{J_3(x)\}^2+\{J_4(x)\}^2+\ldots],$$

as
$$\left.\begin{aligned}\int_0^\pi 2\cos^2 n\phi\,d\phi&=\int_0^\pi (1+\cos 2n\phi)\,d\phi=\pi,\\ \int_0^\pi 2\sin^2 m\phi\,d\phi&=\int_0^\pi (1-\cos 2m\phi)\,d\phi=\pi,\end{aligned}\right\}$$

where m and n are any positive integers,

or $$1=\{J_0(x)\}^2+2\,[\{J_1(x)\}^2+\{J_2(x)\}^2+\{J_3(x)\}^2+\{J_4(x)\}^2+\ldots]$$

Aliter.

$$e^{zx/2}\times e^{-z/2x}=e^{z/2(x-1/x)}$$

$$=\sum_{-\infty}^{\infty} x^n J_n(z)$$

$$=J_0(z)+xJ_1(z)+x^2J_2(z)+\ldots$$
$$+\frac{1}{x}J_{-1}(z)+\frac{1}{x^2}J_{-2}(z)+\ldots$$
$$=J_0(z)+\left(x-\frac{1}{x}\right)J_1(z)+\left(x^3+\frac{1}{x^2}\right)J_2(z)$$
$$+\left(x^2-\frac{1}{x^2}\right)J_3(z)+\ldots \qquad \ldots(A)$$

and $\quad e^{-zx/2}\times e^{z/2x}=e^{z/2(-x+1/x)}=\sum_{-\infty}^{\infty} x^n J_n(z)$

$$=J_0(z)+\left(\frac{1}{x}-x\right)J_1(z)+\left(\frac{1}{x^2}+x^2\right)J_2(z)$$
$$+\left(\frac{1}{x^3}-x^3\right)J_3(z)+\ldots \qquad \ldots(B)$$

Multiplying (A) and (B) and equating coefficients of x^0,

$$1=\{J_0(z)\}^2+2\{J_1(z)\}^2+2\{J_2(z)\}^2+\ldots$$

Ex. 4. *Deduce from the result of example 3*

$: |J_0(x)| \leqslant 1$, and $|J_n(x)| \leqslant 2^{-1/2}$. **(Agra 57)**

From the result of example 3 above,

$$\{J_0(x)\}^2=1-2\,[\{J_1(x)\}^2+\{J_2(x)\}^2+\ldots].$$
$$\therefore \quad \{J_0(x)\}^2 \leqslant 1 \quad \text{or} \quad |J_0(x)| \leqslant 1.$$

Since

$$2\{J_n(x)\}^2=1-2\,[\{J_0(x)\}^2/2+\{J_1(x)\}^2+\{J_2(x)\}^2+\ldots$$
$$\ldots+\{J_{n-1}(x)\}^2+\{J_{n+1}(x)\}^2+\ldots].$$
$$\therefore \quad \{J_n(x)\}^2 \leqslant \frac{1}{2}, \quad \text{or} \quad |J_n(x)| \leqslant \frac{1}{2^{1/2}}, \quad \text{or} \quad \leqslant (2)^{-1/2}.$$

Ex. 5. *From the recurrence formula*

$$2J'_n(x)=J_{n-1}(x)-J_{n+1}(x),$$

deduce the result

$$2^r J^{(r)}(x)=J_{n-r}-rJ_{n-r+2}+\frac{r(r-1)}{2!}J_{n-r+4}-\ldots+(-1)^r J_{n+r}.$$

(Agra 1963, 1958)

Differentiating the result

$$J'_n(x)=\frac{J_{n-1}(x)-J_{n+1}(x)}{2}. \qquad \ldots(1)$$

$$J_n''(x)=\tfrac{1}{2}\{J'_{n-1}(x)-J'_{n+1}(x)\}$$

$$=\frac{1}{2}\left\{\frac{J_{n-2}(x)-J_n(x)}{2}-\frac{J_n(x)-J_{n+2}(x)}{2}\right\}$$

$$=\frac{1}{2^2}\{J_{n-2}(x)-2J_n(x)+J_{n+2}(x)\}$$

$$=\frac{1}{2^2}\{J_{n-2}(x)-{}^2C_1J_n(x)+{}^2C_2J_{n+2}(x)\}. \qquad \ldots(2)$$

$$J_n'''(x)=\frac{1}{2^2}\{J'_{n-2}(x)-2J'_n(x)+J'_{n+2}(x)\}.$$

$$=\frac{1}{2^2}\left[\frac{J_{n-3}(x)-J_{n-1}(x)}{2}-2.\frac{J_{n-1}(x)-J_{n+1}(x)}{2}+\frac{J_{n+1}(x)-J_{n+3}(x)}{2}\right]$$

$$=\frac{1}{2^3}[J_{n-3}(x)-{}^3C_1J_{n-1}(x)+{}^3C_2J_{n+1}(x)-{}^3C_3J_{n+3}(x)].$$

Differentiating in this manner s times,

$$J_n^{(s)}(x)=\frac{1}{2^s}[J_{n-s}(x)-{}^sC_1J_{n-s+2}(x)+{}^sC_2J_{n-s+4}(x)-\ldots$$
$$\ldots+(-1)^s J_{n+s}(x)]. \qquad \ldots(F)$$

Supposing the law indicated in (F) is true for s differentiations; we shall show that, for that reason, it is bound to be true for $s+1$ differentiations.

Differentiating (F) once again,

$$J_n^{(s+1)}(x)=\frac{1}{2^s}[J'_{n-s}(x)-{}^sC_1J'_{n-s+2}(x)+{}^sC_2J'_{n-s+4}(x)-\ldots$$
$$+(-1)^sJ'_{n+s}(x)]$$

$$=\frac{1}{2^s}\left\{\frac{J_{n-s-1}(x)-J_{n-s+1}(x)}{2}\right.$$
$$-{}^sC_1.\frac{J_{n-s+1}(x)-J_{n-s+3}(x)}{2}$$

$$+{}^sC_2\,\frac{J_{n-s+3}(x)-J_{n-s+5}(x)}{2}$$

$$-{}^sC_3\,\frac{J_{n-s+5}(x)-J_{n-s+7}(x)}{2}+\ldots$$

$$\ldots+(-1)^s\,\frac{J_{n+s-1}(x)-J_{n+s+1}(x)}{2}\Bigg\}$$

$$=\frac{1}{2^{s+1}}\,[J_{n-(s+1)}-(1+{}^sC_1)\,J_{n-s+1}+({}^sC_1+{}^sC_2)J_{n-s+3}$$

$$-({}^sC_2+{}^sC_3)\,J_{n-s+5}+\ldots$$

$$\ldots+(-1)^s\,({}^sC_{s-1}+{}^sC_s)\,J_{n+s-1}+(-1)^{s+1}\,J_{n+s+1}]$$

or
$$2^{s+1}J_n^{(s+1)}(x)=J_{n-(s+1)}-{}^{s+1}C_1\,J_{n-s+1}+{}^{s+1}C_2\,J_{n-s+3}$$

$$-{}^{s+1}C_3\,J_{n-s+5}+\ldots+(-1)^s.{}^{s+1}C_s\,J_{n+s-1}$$

$$+(-1)^{s+1}\,J_{n+s+1}. \qquad \ldots(\mathrm{F}+1)$$

From (F) and $(\mathrm{F}+1)$, it is clear that the same law as holds for s differentiations holds good for $(s+1)$ differentiations also. Hence the law is generally true.

Hence, by what is known as mathematical induction, we have

$$2^rJ_n^{(r)}(x)=J_{n-r}-rJ_{n-r+2}+\frac{r\,(r-1)}{2\,!}\,J_{n-r+4}-\ldots+(-1)^rJ_{n+r}$$

whatever integral positive number r may be.

Ex. 6. *Prove that*

$$\frac{J_{n+1}(x)}{J_n(x)}=\frac{x/2}{(n+1)-}\,\frac{(x/2)^2}{(n+2)-}\,\frac{(x/2)^2}{(n+3)-}\ldots$$

[Agra M. Sc. (Final) 1948]

We know $\quad J_{n-1}+J_{n+1}=\dfrac{2n}{x}\,J_n$

or
$$J_{n-1}=\frac{2n}{x}\,J_n-J_{n+1}$$

or
$$J_n=\frac{2\,(n+1)}{x}\,J_{n+1}-J_{n+2} \qquad \ldots(1)$$

[putting $n+1$ for n].

Now $$\frac{J_{n+1}}{J_n}=\frac{1}{\dfrac{J_n}{J_{n+1}}}=\frac{1}{\dfrac{\dfrac{2(n+1)}{x}J_{n+1}-J_{n+2}}{J_{n+1}}}\qquad \text{[from (1)]}$$

$$=\frac{1}{\dfrac{2(n+1)}{x}-\dfrac{J_{n+2}}{J_{n+1}}}=\frac{1}{\dfrac{2(n+1)}{2}-\dfrac{1}{J_{n+1}/J_{n+2}}}$$

$$=\frac{1}{\dfrac{2(n+1)}{x}-\dfrac{1}{\dfrac{\dfrac{2(n+2)}{x}J_{n+2}-J_{n+3}}{J_{n+2}}}}$$

$$=\frac{1}{\dfrac{2(n+1)}{x}-\dfrac{1}{\dfrac{2(n+2)}{x}-\dfrac{1}{J_{n+2}/J_{n+3}}}}$$

$$=\frac{1}{\dfrac{2(n+1)}{x}-\dfrac{1}{\dfrac{2(n+2)}{x}-\dfrac{1}{\dfrac{2(n+3)}{x}-\dots}}}$$

$$=\frac{x/2}{(n+1)-\dfrac{x/2}{\dfrac{2(n+2)}{x}-\dfrac{1}{2(n+3)/x\dots}}}$$

$$=\frac{x/2}{(n+1)-}\;\frac{(x/2)^2}{(n+2)-\dfrac{x/2}{2(n+3)/x-\dots}}$$

$$=\frac{x/2}{(n+1)-}\;\frac{(x/2)^2}{(n+2)-}\;\frac{(x/2)^2}{(n+3)-}\dots$$

Exercise 2 (6)

1. Prove that

$$J_3(x)+3J'_0(x)+4J_0'''(x)=0.$$

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2. Prove that

$$J_1(x)+J_3(x)+J_5(x)+\ldots$$
$$=\tfrac{1}{2}\left[J_0(x)+\int_0^x \{J_0(t)+J_1(t)\}\,dt-1\right].$$

3. Prove that

$$\cos x=J_0(x)-2J_2(x)+2J_4(x)-\ldots$$
$$\sin x=2J_1(x)-2J_3(x)+2J_5(x)-\ldots$$

4. Show that

$$x\sin x=2\,[2^2J_2(x)-4^2J_4(x)+6^2J_6(x)-\ldots],$$
$$x\cos x=2\,[1^2J_1(x)-3^2J_3(x)+5^2J_5(x)-\ldots].$$

5. Show that

$$1=J_0(x)+2J_2(x)+2J_4(x)+\ldots,$$
$$x=2J_1(x)+2.3J_3(x)+2.5.J_5(x)+\ldots$$

[**Hint.** In (1) of Art. 2·41 put $\phi=0$; in (2) of Art. 2·41, divide by ϕ and let ϕ tend to zero.]

6. Show that

$$1=J_0(x)+2J_2(x)+2J_4(x)+\ldots,$$
$$x=2J_1(x)+2.3J_3(x)+2.5J_5(x)+\ldots$$

[**Hint.** For the first result put $\phi=0$ in (1) of Art. 2·14, and for the second result divide (2) of Art. 2·41 by ϕ and then let ϕ tend to zero.]

CHAPTER II

PART 7

2·7. To obtain the second solution of the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0,$$

which is Bessel's equation for $n=0$. **(Agra 1953)**

This is an equation of the second order, and hence it must have two solutions.

Since the two roots of $k^2=0$ in (1) of Art. 2·1 are identical each being equal to zero, we get only one solution $J_0(x)$ of this equation by the method of integration of series in Art. 2·1.

Substitute $y=uJ_0(x)+w$ in the differential equation, where u and w are functions of x. We get

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}+w=-J_0\left(\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}\right)-\frac{2\,du}{dx}\frac{dJ_0}{dx}. \qquad \ldots(1)$$

Now u being at our choice, we take $u=\log x$, so that

$$\frac{d^2u}{dx^2}+\frac{1}{x}\frac{du}{dx}=0.$$

We do this in order to make our equation simpler.

Hence the equation (1) becomes

$$\frac{d^2w}{dx^2}+\frac{1}{x}\frac{dw}{dx}+w=-\frac{2}{x}\frac{dJ_0}{dx}=-\frac{2}{x}[-J_1] \qquad \text{[Exercise 1, 2 (3)]}$$

$$=\frac{2}{x}\left[\frac{2}{x}(2J_2-4J_4+6J_6-\ldots)\right]$$

[by Solved Exercise 1, Chap. II Part 6]

$$=\frac{4}{x^2}\,\Sigma\,(-1)^{(n/2)-1}\,nJ_n(x),\ n \text{ being even.} \quad \ldots(2)$$

Now λJ_n is a particular solution of the equation

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=\lambda\,\frac{n^2}{x^2}\,J_n \qquad \ldots(3)$$

by virtue of Bessel's general equation.

The general term of right hand side of (2) is

$$\frac{4}{x^2}(-1)^{(n/2)-1}\,nJ_n=\frac{n^2}{x^2}\left\{\frac{4\,(-1)^{(n/2)-1}}{n}\right\}J_n\,(x).$$

Hence $\lambda_n=(-1)^{n/2-1}.\frac{4}{n}$, for the general term.

Hence from (3) combined with (2),

$$w=\Sigma\lambda_nJ_n=2\left[J_2-\frac{J_4}{2}+\frac{J_6}{3}-\frac{J_8}{4}+\frac{J_{10}}{5}-\ldots\right].$$

Hence the solution of the original equation is

$$y=uJ_0\,(x)=w$$

or
$$y=J_0\,(x)\log x+2\left[J_2-\frac{J_4}{2}+\frac{J_6}{3}-\ldots\right].$$

This solution is denoted by $Y_0\,(x)$. Hence the primitive of the equation $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$ is $y=AJ_0\,(x)+BY_0\,(x)$.

2·71. To establish the relation

$$J_nJ'_{-n}-J_n'J_{-n}=\frac{-2\sin n\pi}{\pi x}.$$

(Agra 1961, 1959)

We have

$$J_n''+\frac{1}{x}J_n'+\left(1-\frac{n^2}{x^2}\right)J_n=0. \qquad \ldots(1)$$

$$J''_{-n}+\frac{1}{x}J'_{-n}+\left(1-\frac{n^2}{x^2}\right)J_{-n}=0. \qquad \ldots(2)$$

Multiplying (1) by J_{-n} and (2) by J_n and subtracting, we get

$$J_n''J_{-n}-J''_{-n}J_n+\frac{1}{x}(J_n'J_{-n}-J'_{-n}J_n)=0$$

or $$T'+\frac{1}{x}T=0, \text{ where } T=J_n'J_{-n}-J'_{-n}J_n$$

or $$\frac{T'}{T}=-\frac{1}{x}, \quad \text{or} \quad \log T=\log\frac{C}{x}. \qquad \ldots(2)$$

or $$T=J'_nJ_{-n}-J'_{-n}J_n=\frac{C}{x}, \qquad \ldots(3)$$

where C is an arbitrary constant. Comparing coefficients of $\frac{1}{x}$ on both sides of (3), we have

$$\frac{1}{2^n\Gamma(n+1)}\cdot\frac{1}{2^{-n}\Gamma(-n+1)}\{n-(-n)\}=C$$

or $$\frac{2n}{\Gamma(n+1)\ \Gamma(-n+1)}=C=\frac{2}{\Gamma(n)\ \Gamma(1-n)}=\frac{2}{\dfrac{\pi}{\sin n\pi}}$$

$$=\frac{2\sin n\pi}{\pi}.$$

Hence from (3), $T=\dfrac{C}{x}$

or $$J_n'J_{-n}-J'_{-n}J_n=\frac{2\sin n\pi}{\pi x} \qquad \ldots(F_1)$$

or $$J_nJ'_{-n}-J_n'J_{-n}=\frac{-2\sin n\pi}{x\pi} \qquad \ldots(F_2)$$

or $$\frac{J'_{-n}J_n-J_n'J_{-n}}{J_n^2}=\frac{-2\sin n\pi}{\pi x\,.\,J_n^2}$$

or $$\frac{d}{dx}\left\{\frac{J_{-n}(x)}{J_n(x)}\right\}=\frac{-2\sin n\pi}{\pi x\,J_n^2(x)}. \qquad \ldots(F_3)$$

(Agra 1961)

(F_1) and (F_2) and (F_3) are three forms of the same result.

CHAPTER III

PART I

Expansions in Legendre's Polynomials.

3·1. Odd and even functions.

Definition. *$f(x)$ is said to be an* **odd function** *of x, if $f(-x)=-f(x)$.*

Let $f(x)=x^3$; then $f(-x)=(-x)^3=-x^3=-f(x)$;
$f(x)=\sin x$, then $f(-x)=\sin(-x)=-\sin x=-f(x)$;
$f(x)=P_7(x)$, then $f(-x)=P_7(-x)=(-1)^7 P_7(x)$
$=-P_7(x)=-f(x)$.

Thus $\mathbf{x^3}$, sin x, and $\mathbf{P_7(x)}$ are all odd functions of x.

Definition. *$f(x)$ is said to be an even function of x, if $f(-x)=f(x)$.*

Let $f(x)=5x^4+x^2$, then $f(-x)=5(-x)^4+(-x)^2$
$=5x^4+x^2=f(x)$;
$f(x)=\cos x$, then $f(-x)=\cos(-x)=\cos x=f(x)$;
$f(x)=J_0(x)$, then $f(-x)=J_0(-x)=J_0(x)=f(x)$.

Thus $5x^4+x^2$, *$\cos(x)$, and $J_0(x)$ are all even functions.*

3·11. If an even function $f(x)$ is expanded as a series of functions $U_n(x)$, so that

$$f(x)=U_1(x)+U_2(x)+\ldots+U_n(x)+\ldots, \qquad \ldots(1)$$

none of U's can be an odd function of x. For if it were so possible, the right hand side of (1) will change, while the left hand side shall remain unchanged, when x is replaced by $-x$.

A similar remark applies to an odd function of x, *that is in the expansion of an odd function we cannot have auy terms which are even functions of x.*

3·12. Expansion of x^n in Legendre's Polynomials.

As $P_n(x)$ contains terms of degree n and lower, we cannot have in the expansion of x^n any P with suffix higher than n. Thus we assume that

$$x^n = a_n P_n(x) + a_{n-2} P_{n-2}(x) + a_{n-4} P_{n-4}(x) + \ldots, \quad \ldots(1)$$

as $P_n(x)$, $P_{n-2}(x)$, $P_{n-4}(x)$, ... are all even or odd functions according as x^n is an even or odd function. It is also clear that $P_{n-1}(x)$, $P_{n-3}(x)$, $P_{n-5}(x)$,... cannot occur in this expansion, as they are odd functions if x^n is an even function, and even functions if x^n is an odd function.

In order to determine a_r in (1) multiply both sides by $P_r(x)$ and integrate; then by virtue of Art. 1·71,

$$\int_{-1}^{1} x^n P_r(x)\, dx = a_r \int_{-1}^{1} P_r^2(x)\, dx = \frac{2}{2r+1} a_r.$$

Therefore, $a_r = \dfrac{2r+1}{2} \displaystyle\int_{-1}^{1} x^n P_r(x)\, dx$

or, $$a_r = \frac{2r+1}{2} \cdot \frac{1}{2^r r!} \int_{-1}^{1} x^n \left(\frac{d}{dx}\right)^r (x^2-1)^r\, dx \text{ [Art. 1·22].}$$

Integrating by parts, we get

$$a_r = \frac{2r+1}{2^{r+1} r!} \left[\left\{ x^n \left(\frac{d}{dx}\right)^{r-1} (x^2-1)^r \right\}_{-1}^{1} - \int_{-1}^{1} n.x^{n-1} \left(\frac{d}{dx}\right)^{r-1} (x^2-1)^r\, dx \right]$$

or, $$a_r = -\frac{(2r+1)\,n}{2^{r+1} r!} \int_{-1}^{1} x^{n-1} \left(\frac{d}{dx}\right)^{r-1} (x^2-1)^r\, dx$$

[since the first part vanishes for both limits].

Repeating this process the necessary number of times, we get

$$a_r = (-1)^r \cdot \frac{(2r+1)}{2^{r+1}} \frac{n!}{r!\,(n-r)!} \int_{-1}^{1} x^{n-r} (x^2-1)^r\, dx$$

or, $$a_r = (-1)^{2r} \frac{2r+1}{2^{r+1}} \frac{n!}{r!\,(n-r)!} \int_{-1}^{1} x^{n-r} (1-x^2)^r\, dx. \quad \ldots(2)$$

Since r is one of the integers $n, n-2, n-4, \ldots$, so, $n-r$ is one of the numbers $0, 2, 4, 6, \ldots$, so that $n-r$ is an even integer including zero. This renders the integrand in (2) an even function of x so that we have from (2),

$$a_r = \frac{2r+1}{2^{r+1}} \frac{n!}{r!(n-r!)} . 2 . \int_0^1 x^{n-r} (1-x^2)^r \, dx.$$

Now put $x^2 = t$; then

$$a_r = \frac{2r+1}{2^{r+1}} \frac{n!}{r!(n-r)!} \int_0^1 t^{(n-r-1)/2} (1-t)^r \, dt$$

$$= \frac{2r+1}{2^{r+1}} \frac{n!}{r!(n-r)!} \frac{\Gamma\left(\frac{n-r-1}{2}+1\right) \Gamma(r+1)}{\Gamma\left(\frac{n-r-1}{2}+1+r+1\right)}$$

$$= (2r+1) \frac{n!}{(n-r)!} \frac{\Gamma\left(\frac{n-r+1}{2}\right)}{\Gamma\left(\frac{n+r+3}{2}\right)}. \quad [\because \Gamma(r+1) = r!]$$

$$= \frac{(2r+1) . n(n-1)(n-2) \ldots (n-r+1) . \Gamma\left(\frac{n-r+1}{2}\right)}{2^{r+1} \frac{n+r+1}{2} . \frac{n+r-1}{2} . \frac{n+r-3}{2} \ldots \frac{n-r+1}{2} \Gamma\left(\frac{n-r+1}{2}\right)}$$

[cancelling factors and applying successively $\Gamma(n+1) = n\,\Gamma(n)$]

or $$a_r = (2r+1) . \frac{n(n-1)(n-2) \ldots (n-r+2)}{(n+r+1)(n+r-1)(n+r-3) \ldots (n-r+3)} \quad \ldots (3)$$

[by obvious cancellation of factors].

Putting $r = n, n-2, n-4, \ldots$ in (3), we get

$$a_n = \frac{2.3.4 \ldots n}{3.5 \ldots .7(2n+1)} (2n+1)$$

$$a_{n-2} = \frac{4.5 \ldots n}{5.7 \ldots 2n-1} . (2n-3)$$

$$= \frac{2.3.4 \ldots n}{3.5.7 \ldots (2n+1)} \times \frac{3(2n+1)}{2.3} (2n-3)$$

$$=\frac{2.3.4\ldots n}{3.5.7\ldots(2n+1)}\cdot\frac{2n+1}{2}.(2n-3).$$

$$a_{n-4}=\frac{6.7.8\ldots n}{7.9\ldots 2n-3}.(2n-7)$$

$$=\frac{2.3.4\ldots n}{3.5.7\ldots 2n+1}\cdot\frac{3.5\,(2n+1)\,(2n-1)}{2.3.4.5}.(2n-7)$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$

substituting the values of a_n, a_{n-2}, a_{n-4},... in (1), we have

$$x^n=\frac{1.2.3\ldots n}{3.5.7\ldots 2n+1}\Big\{(2n+1)\ P_n(x)+(2n-3).\frac{2n+1}{2}\ P_{n-2}(x)$$
$$+(2n-7).\frac{(2n+1)\,(2n-1)}{2.4}\ P_{n-4}(x)$$
$$+(2n-11)\ \frac{(2n+1)\,(2n-1)\,(2n-3)}{2.4.6}\ P_{n-6}(x)+\ldots\Big\}\quad\ldots(F),$$

the last term in (F) being $\frac{1}{n+1}\ P_0(x)$ or $\frac{3}{n+2}\ P_1(x)$ according as n is even or odd. Here it is good to determine a_0 and a_1 independently and not by the formula (3) for a_r.

3·13. Illustrative Examples.

Ex. 1. *Express x^7 as a series in Legendre's polynomials.*

We know from Art. 3·12, that

$$x^n=\frac{n\,!}{1.3.5\ldots 2n+1}\Big\{(2n+1)\ P_n+(2n-3)\left(\frac{2n+1}{2}\right)P_{n-2}$$
$$+(2n-7)\ \frac{(2n+1)\,(2n-1)}{2.4}\ P_{n-4}$$
$$+(2n-11)\ \frac{(2n+1)\,(2n-1)\,(2n-3)}{2.4.6}\ P_{n-6}+\ldots\Big\}$$

Taking $n=7$, we have

$$x^7=\frac{1.2.3.4.5.6.7}{1.3.5.7.9.11.13.15}\Big\{15P_7+11.\frac{15}{2}\ P_5+7.\frac{15.13}{2.4}\ P_3$$
$$+3.\frac{15.13.11}{2.4.6}\ P_1\Big\}$$

$$=\frac{16}{429}P_7+\frac{8}{39}P_5+\frac{14}{33}P_3+\frac{P_1}{3}.$$

Exercise 3 (1).

Prove the following :—

1. $\mu^0=P_0(\mu)$.

2. $\mu^1=P_1(\mu)$.

3. $\mu^2=\frac{2}{3}P_2(\mu)+\frac{1}{3}P_0(\mu)$.

4. $\mu^3=\frac{2}{5}P_3(\mu)+\frac{3}{5}P_1(\mu)$.

5. $\mu^4=\frac{8}{35}P_4(\mu)+\frac{4}{7}P_2(\mu)+\frac{1}{5}P_0(\mu)$.

6. $\mu^5=\frac{8}{63}P_5(\mu)+\frac{4}{9}P_3(\mu)+\frac{3}{7}P_1(\mu)$.

7. $\mu^6=\frac{16}{231}P_6(\mu)+\frac{24}{77}P_4(\mu)+\frac{10}{21}P_2(\mu)+\frac{1}{7}P_0(\mu)$.

8. $\mu^8=\frac{128}{6435}P_8(\mu)+\frac{64}{495}P_6(\mu)+\frac{48}{143}P_4(\mu)$
$$+\frac{40}{99}P_2(\mu)+\frac{1}{9}P_0(\mu).$$

9. $\mu^9=\frac{128}{12155}P_9(\mu)+\frac{192}{2431}P_7(\mu)+\frac{16}{65}P_5(\mu)$
$$+\frac{56}{143}P_3(\mu)+\frac{3}{11}P_1(\mu).$$

10. $\mu^{10}=\frac{256}{46189}P_{10}(\mu)+\frac{128}{2717}P_8(\mu)+\frac{32}{187}P_6(\mu)$
$$+\frac{48}{143}P_4(\mu)+\frac{50}{143}P_2(\mu)+\frac{1}{11}P_0(\mu).$$

11. If $m<n$, show that $\int_{-1}^{1}\mu^m P_n(\mu)\,d\mu=0$.

12. Show that $\int_{-1}^{1}\mu^n P_n(\mu)\,d\mu=\frac{2^{n+1}(n\,!)^2}{(2n+1)\,!}$.

3·14. General result.

If $f(x)=C_0+C_1x+C_2x^2+\ldots$
$$\ldots+C_nx^n+C_{n+1}x^{n+1}+C_{n+2}x^{n+2}+\ldots \qquad \ldots(1)$$
where the C's are constants and there are either only a finite or infinite number of terms in the expansion, then, by picking out the coefficients, we get

$$f(x)=\sum_0^\infty b_nP_n, \qquad \ldots(2)$$

where $b_n=\dfrac{1.2.3\ldots n}{1.3.5\ldots 2n-1}\left\{C_n+\dfrac{(n+1)(n+2)}{2(2n+3)}C_{n+2}\right.$
$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}C_{n+4}+\ldots\right\}.$$

In $f(x)=C_0+C_1x+C_2x^2+\ldots+C_nx^n+C_{n+1}x^{n+1}+\ldots$, let us collect terms which involve $P_n(x)$ after every power of x is replaced by its expansion in P's

Obviously $x^n, x^{n+2}, x^{n+4}, x^{n+6}, \ldots$, when expanded in P's will give each one term involving $P_n(x)$ and no power of x less than the n^{th} can give any term involving $P_n(x)$.

Thus $C_nx^n, C_{n+2}x^{n+2}, C_{n+4}x^{n+4}, \ldots$ in (1) will give each a term involving $P_n(x)$.

In fact, from the result (F) of Art. 3·12,

$$C_nx^n=C_n\frac{1.2.3\ldots n}{1.3.5\ldots 2n+1}\{(2n+1)P_n+\ldots\},$$

$$C_{n+2}x^{n+2}=C_{n+2}\cdot\frac{1.2.3\ldots n+2}{1.3.5\ldots(2n+5)}\left[(2n+5)P_{n+2}(x)\right.$$
$$\left.+\frac{(2n+1)(2n+5)}{2}P_n(x)+\ldots\right]$$

$$C_{n+4}x^{n+4}=C_{n+4}\frac{1.2.3\ldots n+4}{1.3.5\ldots(2n+9)}\left[(2n+9)P_{n+4}(x)\right.$$

$$+(2n+5).\frac{2n+9}{2}P_{n+2}+(2n+1).\frac{(2n+9)(2n+7)}{2.4}P_n+\ldots\Big].$$

... ...

Substituting these values in (1), we find that the coefficient of $P_n(x)$ in the expansion of $f(x)$ in (1) is

$$b_n=\frac{1.2.3\ldots n}{1.3.5\ldots(2n-1)}\left\{C_n+\frac{(n+1)(n+2)}{2.2(n+3)}C_{n+2}\right.$$
$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}C_{n+4}+\ldots\right\}.$$

Thus $f(x)=\sum\limits_0^\infty b_nP_n(x)$,

$$\text{where } b_n=\frac{1.2.3\ldots n}{1.3.5.2n-1}\left[C_n+\frac{(n+1)(n+2)}{2.(2n+3)}C_{n+2}\right.$$
$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}C_{n+4}+\ldots\right]. \quad \ldots(\text{F})$$

3·15. A special case. *One special case of the above expansion is extremely important and useful.*

We have $f(x)=\dfrac{1}{y-x}=\dfrac{1}{y\left(1-\dfrac{x}{y}\right)}=\dfrac{1}{y}\left(1-\dfrac{x}{y}\right)^{-1}$

$$=\frac{1}{y}\left\{1+\frac{x}{y}+\frac{x^2}{y^2}+\frac{x^3}{y^3}+\ldots+\frac{x^n}{y^n}+\ldots\right\}$$
$$=y^{-1}+y^{-2}x+y^{-3}x^2+\ldots y^{-n-1}x^n$$
$$+y^{-n-2}x^{n+1}+y^{-n-3}x^{n+2}+\ldots$$

Here $C_n=y^{-n-1},\ C_{n+2}=y^{-n-3},\ C_{n+4}=y^{-n-5},\ \ldots \quad \ldots(1)$

Hence from Art. 3·14, $f(x)=\dfrac{1}{y-x}=\sum\limits_0^\infty b_nP_n(x)$,

$$\text{where } b_n=\frac{1.2.3\ldots n}{1.3.5.2n-1}\left\{C_n+\frac{(n+1)(n+2)}{2.(2n+3)}C_{n+2}\right.$$
$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}C_{n+4}+\ldots\right\}$$

$$=\frac{1.2.3\ldots n}{1.3.5\ldots 2n-1}\left\{y^{-n-1}+\frac{(n+1)(n+2)}{2.(2n+3)}y^{-n-3}\right.$$
$$\left.+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)}y^{-n-5}+\ldots\right\}$$

[using (1)]

$=(2n+1)\,Q_n(y)$ [Art. 1·13].

Thus $\dfrac{\mathbf{1}}{\mathbf{y-x}}=\sum\limits_{n=0}^{\infty} b_n P_n(x)=\sum\limits_{n=0}^{\infty} \mathbf{(2n+1)\,P_n(x)\,Q_n(y)}$.

...(F)

CHAPTER IV

Legendre's Function of the Second Kind $Q_n(x)$.

4·1. In Arts. 1·12 and 1·13, we have already obtained a series for $Q_n(x)$ in negative powers of x. We have found there,

$$Q_n(x)=\frac{n\,!}{1.3.5\ldots(2n+1)}\left\{x^{-n-1}+\frac{(n+1)\,(n+2)}{2.(2n+3)}\,x^{-n-3}\right.$$

$$\left.+\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4\,(2n+3)\,(2n+5)}\,x^{-n-5}+\ldots\right\}.$$

4·11. Neumann's Integral for $Q_n(x)$, where $x > 1$.

(Agra 1961)

From Art. 3·15,

$$\frac{1}{x-u}=\sum_{n=0}^{\infty}(2n+1)\,P_n(u)\,Q_n(x), \qquad \ldots(1)$$

where $x > 1$ and $-1 \leqslant u \leqslant 1$.

Multiplying both sides of (1) by $P_n(u)$ and integrating with respect to u between the limits -1 and 1, x being independent of u, we have

$$\int_{-1}^{1}\frac{P_n(u)}{x-u}\,du=\left[(2n+1)\int_{-1}^{1}P_n^{2}(u)\,du\right]Q_n(x) \text{ [Art. 1·71 (a)]}$$

$$=(2n+1).\frac{2}{2n+1}.Q_n(x) \text{ [Art. 1·71 (b)]}.$$

$$\therefore\quad \mathbf{Q_n(x)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(u)\,du}{x-u}}. \qquad \ldots(\mathrm{F})$$

This is **Neumann's Integral.**

4·12. Relations between consecutive Legendre's functions of the second kind and also those between functions of the first and the second kinds.

I. To prove

$$\mathbf{n Q_n(x)-(2n-1)\, x Q_{n-1}(x)+(n-1)\, Q_{n-2}(x)=0.}$$

Making use of Neumann's integral, the left hand side is

$$\frac{1}{2}\int_{-1}^{1}[nP_n(u)-(2n-1)\, xP_{n-1}(u)+(n-1)\, P_{n-2}(u)]\,\frac{du}{x-u}\cdot\,\ldots(1)$$

Now by Art. 1·6, we know that

$$nP_n(u)-(2n-1)\, uP_{n-1}(u)+(n-1)\, P_{n-2}(u)=0$$

or, $$nP_n(u)+(n-1)\, P_{n-2}(u)=(2n-1)\, uP_{n-1}(u).$$

Substituting in (1), the left hand side of the result to be proved, now becomes

$$-\frac{1}{2}(2n-1)\int_{-1}^{1}(x-u)\,\frac{(P_{n-1}(u)\, du}{x-u}$$

$$=-\frac{1}{2}(2n-1)\int_{-1}^{1}P_{n-1}(u)\, P_0(u)\, du$$

$$=0 \text{ [Art. 1·71 (a); } P_0(u) \text{ being } =1].$$

This proves the identity I.

II. $$\frac{\mathbf{dQ_{n+1}}}{\mathbf{d\mu}}-\frac{\mathbf{dQ_{n-1}}}{\mathbf{du}}=\mathbf{(2n+1)\, Q_n}.$$

We have $Q_{n+1}(x)=\frac{1}{2}\int_{-1}^{1}\frac{P_{n+1}(u)\, du}{x-u}$.

$$Q'_{n+1}(x)=-\frac{1}{2}\int_{-1}^{1}\frac{P_{n+1}(u)\, du}{(x-u)^2}$$

$$=-\frac{1}{2}\left[P_{n+1}(u)\,\frac{1}{x-u}\right]_{-1}^{1}+\frac{1}{2}\int_{-1}^{1}\frac{P'_{n+1}(u)\, du}{(x-u)}$$

[by integration by parts].

Also $Q'_{n-1}=-\frac{1}{2}\left[P_{n-1}(u)\,\frac{1}{x-u}\right]_{-1}^{1}+\frac{1}{2}\int_{-1}^{1}P'_{n-1}(u)\,\frac{du}{x-u}$.

Since, on evaluation,

$$-\tfrac{1}{2}\left[P_{n+1}(u)\cdot\frac{1}{x-u}\right]_{-1}^{1}=-\tfrac{1}{2}\left[P_{n-1}(u)\,\frac{1}{x-u}\right]_{-1}^{1},$$

we have from the above

$$Q'_{n+1}(x)-Q'_{n-1}(x)=\frac{1}{2}\int_{-1}^{1}\left[P'_{n+1}(u)-P'_{n-1}(u)\right]\frac{du}{x-u}$$

$$=\frac{1}{2}\int_{-1}^{1}(2n+1)\,\frac{P_n(u)\,du}{x-u}\qquad\text{[Art. 1·62]}$$

$$=(2n+1)\,Q_n(x)\qquad\text{[Art. 4·11]}.$$

Thus $\mathbf{Q'_{n+1}(x)-Q'_{n-1}(x)=(2n+1)\,Q_n(x)}$.

III. $\mathbf{(x^2-1)\,Q'_n(x)=\dfrac{n\,(n+1)}{2n+1}\,[Q_{n+1}(x)-Q_{n-1}(x)]}$.

From Legendre's equation,

$$\frac{d}{dx}\,[(x^2-1)\,Q'_n(x)]=n\,(n+1)\,Q_n(x).$$

$$\therefore\quad\left[(x^2-1)\,Q'_n(x)\right]_{\infty}^{x}=n\,(n+1)\int_{\infty}^{x}Q_n(x)\,dx\ldots.(1)$$

From II, we have by integration,

$$(2n+1)\int_{\infty}^{x}Q_n(x)\,dx=\left[Q_{n+1}(x)-Q_{n-1}(x)\right]_{\infty}^{x}.\ \ldots.(2)$$

Hence from (1) and (2),

$$\left[(x^2-1)\,Q'_n(x)\right]_{\infty}^{x}=\frac{n\,(n+1)}{2n+1}\left[Q_{n+1}(x)-Q_{n-1}(x)\right]_{\infty}^{x}\ldots(3)$$

From the series for $Q_n(x)$ in Art. 1·13, we find that for $x=\infty$,

$Q_n(x)=0$, $Q_{n-1}(x)=0$, $Q_{n+1}(x)=0$, $Q'_n(x)=0$, $x^2Q'_n(x)=0$.

Hence from (3),

$$\mathbf{(x^2-1)\,Q_n'(x)=\frac{n\,(n+1)}{2n+1}\,[Q_{n+1}(x)-Q_{n-1}(x)].}$$

IV. (A) $\mathbf{P_n(x)\,Q_n'(x)-Q_n(x)\,P_n'(x)=\dfrac{-1}{x^2-1}}$.

(B) $\mathbf{Q_n(x)=P_n(x)\displaystyle\int_x^{\infty}\frac{dx}{(x^2-1)\,P_n^2(x)}}$. **(Agra 1962)**

From Legendre's equation, we have

$$(x^2-1)\frac{d^2P_n}{dx^2}+2x\frac{dP_n}{dx}-n(n+1)P_n=0, \quad \ldots(1)$$

$$(x^2-1)\frac{d^2Q_n}{dx^2}+2x\frac{dQ_n}{dx}-n(n+1)Q_n=0. \quad \ldots(2)$$

Multiplying (1) by Q_n. and (2) by P_n and subtracting, we have

$$(x^2-1)\left[\frac{d^2P_n}{dx^2}Q_n-\frac{d^2Q_n}{dx^2}P_n\right]+2x\left[\frac{dP_n}{dx}Q_n-\frac{dQ_n}{dx}P_n\right]=0$$

or
$$\frac{d}{dx}\left[(x^2-1)\left(\frac{dP_n}{dx}.Q_n-\frac{dQ_n}{dx}P_n\right)\right]=0$$

or
$$(x^2-1)\left(\frac{dP_n}{dx}Q_n-\frac{dQ_n}{dx}P_n\right)=C$$

or
$$\frac{dP_n}{dx}Q_n-\frac{dQ_n}{dx}P_n=\frac{C}{x^2-1}=\frac{C}{x^2}\left\{1-\frac{1}{x^2}\right\}^{-1} \quad \ldots(3)$$

Substituting in (3), their equivalents in series for $P_n(x)$, $P_n'(x)$, $Q_n(x)$, $Q_n'(x)$ from Arts. 1·13, and then equating the coefficients of $\frac{1}{x^2}$ which is the highest power of x on both sides of (3), we get

$$\frac{1.3.5\ldots(2n-1)}{n!}\cdot\frac{n!}{1.3.5\ldots(2n+1)}[n+(n+1)]=C$$

or
$$1=C.$$

Substituting in (3), we have

$$\frac{dP_n}{dx}Q_n-\frac{dQ_n}{dx}P_n=\frac{1}{x^2-1}.$$

or
$$\mathbf{Q_n'P_n-P_n'Q_n=\frac{-1}{x^2-1}} \quad \ldots(\mathrm{F_1})$$

or
$$\frac{Q_n'(x)P_n(x)-P_n'(x)Q_n(x)}{P_n^2(x)}=-\frac{1}{(x^2-1)P_n^2(x)}$$

or
$$\frac{d}{dx}\left[\frac{Q_n(x)}{P_n(x)}\right]=-\frac{1}{(x^2-1)P_n^2(x)}$$

or
$$\left[\frac{Q_n(x)}{P_n(x)}\right]_x^\infty=-\int_x^\infty \frac{dx}{(x^2-1)\,P_n^2(x)} \qquad \ldots(4)$$
or
$$-\frac{Q_n(x)}{P_n(x)}=-\int_x^\infty \frac{dx}{(x^2-1)\,P_n^2(x)}$$
[as the ieft side of (4) $=0$ for the upper limit]

or
$$\mathbf{Q_n(x)=P_n(x)\int_x^\infty \frac{dx}{(x^2-1)\,P_n^2(x)}}. \qquad \textbf{(Agra 1962)}$$

V. $\mathbf{(P_nQ_{n-1}-Q_nP_{n-1})=\frac{1}{n}}$.

We have
$$nP_n+(n-1)\,P_{n-2}=(2n-1)\,xP_{n-1}, \qquad \text{[Art. 1·6]}$$
$$nQ_n+(n-1)\,Q_{n-2}=(2n-1)\,xQ_{n-1}. \qquad \text{[Art. 4·12]}$$

Multiplying the first by Q_{n-1} and the second by P_{n-1} and subtracting, we have
$$n\,(P_nQ_{n-1}-P_{n-1}Q_n)+(n-1)\,(Q_{n-1}P_{n-2}-Q_{n-2}P_{n-1})=0$$
or $$n\,(P_nQ_{n-1}-Q_nP_{n-1})=(n-1)\,(P_{n-1}Q_{n-2}-Q_{n-1}P_{n-2})$$

Hence, we have
$$f_n=f_{n-1}=f_{n-2}=\ldots=f_1,$$
where f_n stands for $n\,(P_nQ_{n-1}-Q_nP_{n-1})$

Thus $$n\,(P_nQ_{n-1}-Q_nP_{n-1})=P_1Q_0-Q_1P_0. \qquad \ldots(1)$$

Now $$Q_1=P_1(x)\int_x^\infty \frac{dx}{(x^2-1)\,[P_1(x)]^2} \quad \text{[by IV]}$$
$$=x\int_x^\infty \frac{dx}{(x^2-1)\,x^2} \qquad \ldots(2)$$
[as $P_1(x)=x$].

$$Q_0=P_0\int_x^\infty \frac{dx}{(x^2-1)\,P_0^2}=\int_x^\infty \frac{dx}{(x^2-1)}, \text{ as } P_0=1.$$
$\ldots(3)$

Now $$Q_1=x\int_x^\infty \frac{dx}{(x^2-1)\,x^2}=x\int_x^\infty \frac{dx}{x^2-1}-x\int_x^\infty \frac{dx}{x^2}$$
$$=xQ_0-1 \qquad \ldots(4)$$
$$\left[\text{from (3) as } x\int_x^\infty \frac{dx}{x^2}=1\right].$$

Thus $P_1Q_0-Q_1P_0=xQ_0-(xQ_0-1).1$ [from (4), as $P_0=1$]

$$=1 \qquad \ldots(5)$$

so that from (1) and (5),

$$n\,(P_nQ_{n-1}-Q_nP_{n-1})=1$$

or

$$\mathbf{P_nQ_{n-1}-Q_nP_{n-1}=\frac{1}{n}}. \qquad \ldots(\mathrm{F})$$

VI. $\mathbf{P_nQ_{n-2}-Q_nP_{n-2}=\dfrac{(2n-1)}{n\,(n-1)}\,x}$.

We have, $nP_n+(n-1)\,P_{n-2}=(2n-1)\,x\,P_{n-1}$,

$$nQ_n+(n-1)\,Q_{n-2}=(2n-1)\,x\,Q_{n-1}.$$

Multiplying the first by Q_{n-2} and the second by P_{n-2} and subtracting, we have

$$n\,(P_nQ_{n-2}-Q_nP_{n-2})=(2n-1)\,x\,(P_{n-1}Q_{n-2}-Q_{n-1}P_{n-2})$$

$$=\frac{2n-1}{n-1}\,x \quad \text{[from V]}.$$

Therefore, $\mathbf{P_nQ_{n-2}-Q_nP_{n-2}=\dfrac{2n-1}{n\,(n-1)}\,x}$ $\qquad \ldots(\mathrm{F})$

4·13. Solved Example.

Show that

$$\frac{d^{n+1}Q_n}{dx^{n+1}}=-\frac{(-2)^n\,n\,!}{(x^2-1)^{n+1}}.$$

We know

$$Q_n(x)=\frac{n\,!}{1.3.5\ldots(2n+1)}\left[x^{-n-1}+\frac{(n+1)\,(n+2)}{2.(2n+3)}\,x^{-n-3}\right.$$
$$\left.+\frac{(n+1)\,(n+2)\,(n+3)\,(n+4)}{2.4\,(2n+3)\,(2n+5)}x^{-n-5}+\ldots\right].$$

Differentiating $(n+1)$ times,

$$\frac{d^{n+1}Q_n(x)}{dx^{n+1}}=\frac{n\,!\,(-1)^{n+1}}{1.3.5\ldots(2n+1)}\left[(n+1)\,(n+2)\ldots(2n+1)\,x^{-2n-2}\right.$$
$$+\frac{(n+1)\,(n+2)\ldots(2n+3)}{2.(2n+3)}x^{-2n-4}$$
$$\left.+\frac{(n+1)\,(n+2)\ldots(2n+5)}{2.4.(2n+3)\,(2n+5)}x^{-2n-6}+\ldots\right]$$

$$=\frac{(-1)^{n+1}x^{-2n-2}}{1.3.5\ldots(2n+1)}\Big[(2n+1)\,!+\frac{(2n+2)\,!}{2}x^{-2}$$

$$+\frac{(2n+2)\,!}{2.4}.(2n+4)x^{-4}+\frac{(2n+2)\,!}{2.4.6}.(2n+4)(2n+6)x^{-6}$$

$$+\ldots\Big]$$

$$=\frac{(-1)^{n+1}x^{-2n-2}(2n+1)\,!}{1.3.5\ldots(2n+1)}\Big[1+(n+1)\frac{1}{x^2}$$

$$+\frac{(n+1)(n+2)}{1.2}\frac{1}{x^4}+\frac{(n+1)(n+2)(n+3)}{1.2.3}.\frac{1}{x^6}+\ldots\Big]$$

$$=\frac{(-1)^{n+1}2^n(n\,!).(2n+1)\,!\,x^{-2n-2}}{1.3.5\ldots(2n+1).2^n.n\,!}\left[1-\frac{1}{x^2}\right]^{-n-1}$$

$$=-\frac{(-2)^n.(n\,!)\,(2n+1)\,!}{(2n+1)\,!\,x^{2n+2}}\left[\frac{x^2-1}{x^2}\right]^{-n-1}$$

$$=-\frac{(-2)^n.n\,!}{x^{2n+2}.x^{-2n-2}}.(x^2-1)^{-n-1}$$

$$=-\frac{(-2)^n.n\,!}{(x^2-1)^{n+1}}.$$

Exercise 4 (1)

1. Show that

$$Q_n(\mu)=2^n.n\,!\int_\mu^\infty d\mu\int_\mu^\infty d\mu\ldots\int_\mu^\infty(\mu^2-1)^{-n-1}\,d\mu,$$

the integrations being carried out $(n+1)$ times.

$$\Big[(\mu^2-1)^{-n-1}=\frac{1}{\mu^{2n+2}}+\frac{n+1}{\mu^{2n+4}}+\frac{(n+1)(n+2)}{2\,!\,\mu^{2n+6}}+\ldots$$

where $|\mu|>1$, integrate both sides with respect to μ between the limits μ to ∞, $(n+1)$ times.]

2. $Q'_{n+1}-xQ_n'=(n+1)\,Q_n.$
[Differentiate I of Art. 4·12 and eliminate Q'_{n-1} between the result and II of Art. 4·12.]

3. $x\dfrac{dQ_n}{dx}-\dfrac{dQ_{n-1}}{dx}=nQ_n.$

[Subtract the result of question 2, from that of II of Art. 4·12.]

4. $(x^2-1)\dfrac{dQ_n}{dx}=nxQ_n-nQ_{n-1}.$

[Multiply the result of question 3 by x, and write $n-1$ for n in the result of question 2 and subtract the latter result from the former.]

AGRA UNIVERSITY EXAMINATION PAPERS

1952

1. Define P_n and show that

(i) $\dfrac{dP_n}{dx}-\dfrac{dP_{n-1}}{dx}=(2n-1)\,P_{n-1}$.

(ii) $\displaystyle\int_{-1}^{1} P_m(x)\,P_n(x)=0,$

where m and n are different positive integers and that

$$\int_{-1}^{1} P_n^2(x)\,dx=\frac{2}{2n+1}.$$

2. Define $J_n(x)$ and show that

$$J_{n-1}+J_{n+1}=\frac{2}{x}\,nJ_n.$$

Prove that when n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of $e^{x\,(z-1/z)/2}$ in ascending and descending powers of z and can be expressed as

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos\,(n\theta-x\sin\theta)\,d\theta.$$

3. Define a linear partial differential equation and explain Lagrange's Method of solving it.

(i) Solve $(t+y+z)\dfrac{\partial t}{\partial x}+(t+x+z)\dfrac{\partial t}{\partial y}$

$$+(t+x+y)\frac{\partial t}{\partial z}=(x+y+z).$$

(ii) Solve $\dfrac{\partial^2 z}{\partial x^2}+3\,\dfrac{\partial^2 z}{\partial x\,\partial y}+2\,\dfrac{\partial^2 z}{\partial y^2}=x+y.$

4. Explain Monge's method of integrating the equation $Rr+Ss+Tt=V$.

Apply this method to integrate the equation

$$ar+bs+ct+e\ (rt-s^2)=h,$$

where a, b, c, e and h are constants.

1953

1. (a) Define P_n and show that

$$nP_n=(2n-1)\ xP_{n-1}-(n-1)\ P_{n-2}.$$

(b) Prove that $P_n=\frac{1}{\pi}\int_0^{\pi}\{x\pm(x^2-1)^{1/2}\cos\theta\}^n\,d\theta$, n being a positive integer throughout.

2. (a) Solve the equation $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+y=0$.

(b) Define J_n, n being a positive integer, and prove that

$$J_{n+3}+J_{n+5}=\frac{2}{x}\ (n+4)\ J_{n+4}.$$

3. (a) Define a linear partial differential equation and give its geometrical meaning.

Solve the equation $(mz-ny)\ p+(nx-lz)q=ly-mx$.

(b) Solve $x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}-y\frac{\partial z}{dy}+x\frac{\partial z}{\partial x}=0$.

4. Explain Monge's method of integration of the equation $Rr+Ss+Tt+U(rt-s^2)=V$.

Solve the equation

$$z(1+q^2)r-2pqzs+z(1+p^2)t-z^2(s^2-rt)+1+p^2+q^2=0.$$

1954

1. Solve the equations :—

(i) $(y^3x-2x^4)\ p+(2y^2+x^3y)\ q=2z\ (x^3-y^3)$.

(ii) $$\frac{\partial^2 z}{\partial x^2}-\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}-2\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}$$
$$=e^{2x+3y}+\sin\ (2x+y)+xy.$$

2. Explain Monge's method of integration of the equation $Rr+Sz+Tt=V$.

Apply this method to integrate the equation

$$(x-y)\,(xr-xs-ys+yt)=(x+y)\,(p-q).$$

3. Show that P_n is the coefficient of z^n in the expansion in ascending powers of z of $(1-2xz+z^2)^{-1/2}$.

Hence or otherwise show that

(i) $(2n+1)\,xP_n=(n+1)\,P_{n+1}+nP_{n-1}$,

and (ii) $nP_n=xP'_n-P'_{n-1}$.

4. Define J_n and show that

(i) $xJ'_n=nJ_n-xJ_{n+1}$,

and (ii) $xJ'_n=-nJ_n+xJ_{n-1}$,

where $J_n'=\dfrac{dJ_n}{dx}$.

1955

1. (a) Solve :—

(i) $$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 u}{\partial z^3}-\frac{3\partial^3 u}{\partial x\,\partial y\,\partial z}=x^3-3xyz.$$

(ii) $$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial z}{\partial y}-z=\cos\,(x+2y)+e^y.$$

(b) Apply Monge's method of integration to solve the equation $r+(a+b)s+abt=xy$.

2. (a) Define P_n and show that

$$\frac{dP_n}{dx}-\frac{dP_{n-2}}{dx}=(2n-1)\,P_{n-1}.$$

(b) Show that $\int_{-1}^{1} P_m(x)\,P_n(x)\,dx=0$, where m and n are different positive integers; and that

$$\int_{-1}^{1} P_n^2(x)dx=\frac{2}{2n+1}.$$

3. (a) Show that $y=\frac{1}{\pi}\int_0^{\pi} \cos (x \cos \phi)\, d\phi$ satisfies the equation $$\frac{d^2y}{dx^3}+\frac{1}{x}\frac{dy}{dx}+y=0.$$

(b) Define J_n, n being a positive integer, and prove that
$$\frac{dJ_n}{dx}=\frac{2}{x}\left\{\frac{1}{2}\,nJ_n-(n+2)\,J_{n+2}+(n+4)\,J_{n+4}-\ldots ad\ inf.\right\}$$

1956

1. (a) Expain clearly Monge's method of solving the equation $Rr+Ss+Tt=V$.

Integrate the equation
$$(b+cq)^2r-2(b+cq)(a+cq)s+(a+cp)^2t=0.$$

(b) Solve
$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x.\partial y}-2\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}=e^{2x+3y}+\sin (2x+y)+xy.$$

2. (a) If $P_n(x)$ is a solution of Legendre's differential equation, prove the formula
$$2^n.n\,!\,P_n(x)=\frac{d^n}{dx^n}\,[(x^2-1)^n].$$

(b) Show that

(i) $$\frac{dP_n(x)}{dx}=(2n-1)P_{n-1}+(2n-5)P_{n-3}+(2n-9)P_{n-5}+\ldots$$

(ii) $$\int_{-1}^{1}\left(\frac{dP_n}{dx}\right)^2 dx=n\,(n+1).$$

3. Prove with the usual notation :—

(i) $J_n(x)=\frac{1}{\pi}\int_0^{\pi} \cos (n\phi-x \sin \phi)\, d\phi.$

(ii) $2n\,J_n=x(J_{n-1}+J_{n+1}).$

(iii) $\frac{1}{2}xJ_n=(n+1)\,J_{n+1}-(n+3)J_{n+3}+(n+5)J_{n+5}-\ldots$

1957

1. (a) Solve :—

(i) $\dfrac{\partial^2 z}{\partial x^2}-\dfrac{\partial^2 z}{\partial y^2}+\dfrac{\partial z}{\partial x}+3\,\dfrac{\partial z}{\partial y}-2z=e^{x-y}-x^2y;$

(ii) $x^2\,\dfrac{\partial^2 z}{\partial x^2}-4yx\,\dfrac{\partial^2 z}{\partial x\,\partial y}+4y^2\,\dfrac{\partial^2 z}{\partial y^2}+6y\,\dfrac{\partial z}{\partial y}=x^3y^4.$

(b) Apply Monge's method of integration to solve the equation $q\,(1+q)r-(p+q+2pq)s+p(1+p)t=0.$

2. Prove that $(1-2xz+z^2)^{-1/2}$ is a solution of the equation

$$z\,\frac{\partial^2\,(zv)}{\partial z^2}+\frac{\partial}{\partial x}\left\{(1-x^2)\,\frac{\partial v}{\partial x}\right\}=0.$$

Hence, or otherwise, prove that

$\dfrac{dP_n}{dx}=(n+1)\,\dfrac{P_{n+1}-xP_n}{x^2-1}$, and $c+\displaystyle\int P_n\,dx=\dfrac{P_{n+1}-P_{n-1}}{2n+1}$,

where P_n is Legendre's function of the nth order.

3. (a) Show that for the Bessel function $J_n\,(z)$,

$$\frac{d}{dz}\,\{z^n\,J_n\,(z)\}=z^n\,J_{n-1}\,(z).$$

(b) Prove that

$$\{J_n\,(z)\}^3+2[\{J_1\,(z)\}^2+2\,\{J_2\,(z)\}^2+\ldots]=1.$$

Deduce that $|\,J_0\,(x)\,|\leqslant 1,\,|\,J_n\,(z)\,|\leqslant 2^{-1/2},\ (n\geqslant 1)$, when z is real.

1958

1. Solve :—

(a) $r+(a+b)s+abt=xy.$

(b) $(D^2-D'^2-3D+3D')\,z=xy+e^{x+2y}.$

c) $\dfrac{1}{x^2}\,\dfrac{\partial^2 z}{\partial x^2}-\dfrac{1}{x^2}\,\dfrac{\partial z}{\partial x}=\dfrac{1}{y^2}\,\dfrac{\partial^2 z}{\partial y^2}-\dfrac{1}{y^2}\,\dfrac{\partial z}{\partial y}.$

2. Obtain Christoffel's summation formula for the sum of the series $\displaystyle\sum_0^n\,(2r+1)P_r\,(x)P_r\,(y).$

Hence prove that

$$(P_0)^2+3P_1^2+5P_2^2+\ldots+(2n+1)\,P_n^2$$
$$=(n+1)^2\,P_n^2-(x^2-1)\,(P'_n)^2.$$

3. Investigate the solution $J_n(x)$ of the Bessel's differential equation in the form

$$J_n(x)=(-2)^n\,x^n\,\frac{d^n}{d\,(x^2)^n}\,J_0(x).$$

Obtain the recurrence formula

$$2J_n'(x)=J_{n-1}(x)-J_{n+1}(x)$$

and deduce that

$$2^r\,J_n^{(r)}(x)=J_{n-r}-r\,J_{n-r+2}+\frac{r\,(r-1)}{2\,!}\,J_{n-r+4}-\ldots+(-1)^r\,J_{n+r}.$$

Prove that

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi}\cos\,(n\phi-x\sin\phi)\,d\phi.$$

1959

1. Solve :—

(i) $r=a^2t.$

(ii) $(b+cq)^2r-2(b+cq)\,(a+cp)s+(a+cp)^2t=0.$

2. Prove in the usual notation, that

(i) $$\frac{1-z^2}{(1-\mu z+z^2)^{3/2}}=\sum_{n=0}^{\infty}(2n+1)z^nP_n(\mu).$$

(ii) $(x^2-1)\,P'_n(x)=nxP_n(x)-nP_{n-1}(x).$

3. (a) Prove in the usual notation, that

$$J_nJ'_{-n}-J_n'J_{-n}=-\frac{2\sin n\pi}{x\pi}.$$

(b) Prove that two linearly independent solution of Bessel's equation cannot have any common zero except possibly $x=0$.

1960

1. (a) Find the second solution of Bessel's equation when n is a positive integer.

(b) Prove that

$$\pi J_n(x)=\int_0^\pi \cos(n\theta-x\sin\theta)\,d\theta.$$

2. (a) Prove that if n is an integer

$$2^n(n\,!)\,P_n(x)=\left(\frac{d}{dx}\right)^n (x^2-1)^n.$$

(b) Prove that the equation $P_n(x)=0$ has n real roots, all being between -1 and $+1$.

3. (a) Solve $4(r-s)+t=16\log(x+2y)$.

(b) Find a surface satisfying

$$2x^2r-5xys+2y^2t+2(px+qy)=0$$

and touching the hyperbolic paraboloid $z=x^2-y^2$ along its section by the plane $y=1$.

1961

1. (a) Solve $r+s-6t=y\cos x$.

(b) Obtain the integral of $q^2r-2pqs+p^2t=0$ in the form $y+xf(z)=F(z)$ and show that this represents a surface generated by straight lines that are parallel to a fixed plane.

2. (a) Show that

$$x(2n+1)P_n=(n+1)P_{n+1}+nP_{n-1},$$
$$(2n+1)P_n=P'_{n+1}-P'_{n-1}.$$

(b) Prove that

$$2Q_n(y)=\int_{-1}^{+1}\frac{P_n(x)}{y-x}\,dx.$$

3. (a) Show that

$$\frac{x}{2}J_{n-1}=nJ_n-(n+2)J_{n+2}+(n+4)J_{n+4}-\dots$$

(b) Prove that

$$\frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right)=-\frac{2\sin n\pi}{\pi x J_n^2}.$$

1962

1. Solve :—

(i) $r=a^2t$,

(ii) $(b+cq)^2 r-2(b+cq)(a+cp)s+(a+cp)^2 t=0$.

2. Prove that

(i) $\int_{-1}^{1} \{P_n(x)\}^2\, dx=\frac{2}{2n+1}$.

(ii) $\int_{-1}^{1} x^2 P_{n+1}(x)\, P_{n-1}(x)\, dx=\frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

3. (a) If n is a positive integer, show that

$$Q_n(x)=P_n(x)\int_x^{\infty} \frac{dx}{(x^2-1)\{P_n(x)\}^2}.$$

(b) Show that when n is a positive integer,

$$J_n(x)=\frac{1}{\pi}\int_0^{\pi} \cos(n\phi-x\sin\phi)\, d\phi.$$

1963

1. (a) Find the surface passing through the two lines $z=x=0$ and $z-1=x-y=0$ and satisfying the differential equation $r-4s+4t=0$.

(b) Solve :—

(i) $r+s-6t=y\cos x$.

(ii) $$\frac{\partial^2 z}{\partial x^2}-\frac{\partial z}{\partial y}\cdot\frac{\partial z}{\partial x}-2\frac{\partial^2 z}{\partial y^2}+2\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}=e^{2x+3y}+\sin(2x+y)+xy.$$

2. Explain Monge's method of integration of the equation

$$Rr+Ss+Tt=V.$$

Apply this method to integrate the equation

$$(x-y)(xr-xs-ys+yt)=(x+y)(p-q).$$

3. (a) Prove that when n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of $e^{x(z-1/z)/2}$ in ascending and descending powers of z.

(b) Obtain the recurrence formula

$$2J_n' = J_{n-1} - J_{n+1}$$

deduce that

$$2^r J_n^{(r)} = J_{n-r} - r\,J_{n-r+2} + \frac{r\,(r-1)}{2\,!} J_{n-r+4} + \ldots + (-1)^r J_{n+r}.$$

(c) Prove that $J_{\frac{1}{2}}(x) = \sqrt{\left(\frac{2}{n\pi}\right)} \sin x.$

4. (a) If $P_n(x)$ is a solution of Legendre's differential equation, prove that

$$2^n . n\,!\, P_n(x) = \frac{d^n}{dx^n}(x^2-1)^n.$$

(b) Prove that

(i) $P'_{n+1} + P_n' = P_0 + 3P_1 + \ldots + (2n+1)\,P_n.$

(ii) $\dfrac{1+z}{z\sqrt{(1-2xz+z^2)}} - \dfrac{1}{z} = \sum_{n=0}^{\infty} (P_n + P_{n+1})\,z^n.$

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